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Complex Geometry of Matrix Models

L. Chekhov[a]

Steklov Mathematical Institute, ITEP, and *LIFR, Moscow, Russia*

A. Marshakov[b] **A. Mironov**[c]

Lebedev Physics Institute and *ITEP, Moscow, Russia*

D.Vasiliev[d]

MIPT, Moscow, Russia and *ITEP, Moscow, Russia*

The paper contains some new results and a review of recent achievements, concerning the multisupport solutions to matrix models. In the leading order of the 't Hooft expansion for matrix integral, these solutions are described by quasiclassical or generalized Whitham hierarchies and are directly related to the superpotentials of four-dimensional $\mathcal{N} = 1$ SUSY gauge theories. We study the derivatives of tau-functions for these solutions, associated with the families of Riemann surfaces (with possible double points), and relations for these derivatives imposed by complex geometry, including the WDVV equations. We also find the free energy in subleading order of the 't Hooft expansion and prove that it satisfies certain determinant relations.

Recent interest to matrix models and especially to their so-called multisupport (multicut) solutions was inspired by the studies in $\mathcal{N} = 1$ SUSY gauge theories due to Cachazo, Intrilligator and Vafa [1], [2] and by the proposal of Dijkgraaf and Vafa [3] to calculate the low energy superpotentials, using the partition function of multicut solutions. The solutions themselves are well-known already for a long time (see, e.g., [4, 5]) with a new vim due to the paper by Bonnet, David and Eynard [6].

The Dijkgraaf–Vafa proposal was to consider the nonperturbative superpotentials of $\mathcal{N} = 1$ SUSY gauge theories in four dimensions (possibly coming as the softly broken $\mathcal{N} = 2$ Seiberg–Witten (SW) theories [7, 8]) arising from the partition functions of the one-matrix model (1MM) in the leading order in $1/N$, N being the matrix size. The leading order (of the 't Hooft $1/N$-expansion) of the matrix model is described by the quasiclassical tau-function of the so-called universal Whitham hierarchy [9] (see also [10, 11, 12, 13, 14], the details about one-matrix and two-matrix cases see in [15] and [16]). One would expect the existence of the relation between the partition function in the planar limit and the quasiclassical, or Whitham hierarchy already because matrix integrals are tau-functions of the hierarchies of integrable equations of the KP/Toda type [17]. For the planar single-cut solutions the matrix model, partition functions become tau-functions of the dispersionless Toda hierarchy, one of the simplest example of the Whitham hierarchy.

One may also consider more general solutions to matrix models, identifying them with generic solutions to the loop (Schwinger–Dyson, or Virasoro) equations [18], to be the Ward identities satisfied by matrix integrals [19]. These generalized solutions to the loop equations can be still treated as

[a]E-mail: chekhov@mi.ras.ru
[b]E-mail: mars@lpi.ru; mars@itep.ru
[c]E-mail: mironov@itep.ru; mironov@lpi.ru
[d]E-mail: vasiliev@itep.ru

matrix model partition functions (e.g., within a D-module ideology [20, 21, 22]); however, they do not necessarily admit any matrix integral representation.

In what follows, we aim to discuss an interesting class of multi-cut, or multi-support, solutions to the loop equations that *have* multi-matrix integral representation [6, 23, 20, 22]. These solutions are associated with families of Riemann surfaces and form a sort of a basis in the space of all solutions to the loop equations [20, 22] (like the finite-gap solutions form a basis in the space of all solutions to an integrable hierarchy). They can be distinguished by their "isomonodromic" properties—switching on higher matrix model couplings, or $1/N$-corrections does not change the family of Riemann surfaces, but just reparameterizes the moduli as functions of these couplings. This property is directly related to that the partition functions of these solutions are quasiclassical tau-functions (also often called as prepotentials of the corresponding Seiberg–Witten-like systems).

In sect. 1, we describe the general properties of multi-cut solutions of matrix models. In particular, we prove that the free energy of the 1MM in the planar (large N) limit coincides with the prepotential of some Seiberg–Witten-like theory. This free energy is the logarithm of a quasiclassical tau-function. The corresponding quasiclassical hierarchy is explicitly constructed.

The Whitham hierarchy is basically formulated in terms of Abelian differentials on a family of Riemann surfaces [9]. This implies the main quantities in matrix models are to be expressed in geometric terms and allow calculating derivatives of the matrix model free energy. Indeed, we demonstrate in sect. 2 that the second derivatives of the logarithm of the matrix model partition function can be expressed through the so-called Bergmann bi-differential.

In sect. 3, we turn to the third derivatives of partition function and to the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [24, 25, 26], which are differential equations, involving the third derivatives of tau-function with respect to Whitham times. These equations are usually considered an evidence for existing an underlying topological string theory. In sect. 3, we prove that the quasiclassical tau-function of the multi-support solutions to matrix models satisfies the WDVV equations in the case of general 1MM solution, i.e., in the case of arbitrary number of nonzero times, which include now, besides the times of the potential, the occupation numbers (filling fractions) indicating the portions of eigenvalues of the corresponding model that dwell on the related intervals of eigenvalue supports. Although being, at first glance, quantities of very different nature in comparison with the original times, they can be nicely combined into a unified set of the "small phase space" of the model [1]. This completes an interpretation of the results of [1, 2] in terms of quasiclassical hierarchies.

The WDVV equations are a simple consequence of the residue formula and associativity of some algebra (e.g., of the holomorphic differentials on the Riemann surface) [26, 27, 28] (see also [29]); moreover, the associativity of algebra can be replaced by a simple counting argument [30]: the number of critical points in the residue formula should be equal to the dimension of the "small phase space". We present the proof of the residue formula and extra conditions following [31]. Note that, while the residue formula is always present in the theories of such type [9, 28],[2] the associativity (naive counting) is often violated [28] ([30]) (maybe the most notorious case is the SW system associated with the elliptic Calogero–Moser model, where this phenomenon was first found in [28]).

There are strong indications that the correspondence between matrix models and SUSY gauge theories goes beyond just the large-N limit of matrix models. Say, the relation between gauge theories and matrix model were further verified at the nonplanar, genus-one level for the solution with two cuts and a cubic matrix model potential [23]. In sect. 4, we find the multicut solution to 1MM in the subleading order: the torus approximation in terms of a dual string theory. We calculate the corresponding free energy and, in particular, prove it to have a determinant form, related to the

[1] Note that, from the point of view of $\mathcal{N}=1$ SUSY theory, couplings in the matrix model potential must be identified with couplings in the tree superpotential, while the occupation numbers play the role of moduli of the loop equation solutions and must be associated with the vacuum expectation values of the gluino condensates.

[2] The recently proposed in [32] residue formulas in planar case just follow from the standard residue formula. However, the residue formulas of [32] involving non-planar corrections look new and rather instructive.

topological B-model on the local geometry $\widehat{II}$ and conjectured in [3] (the authors of [23] have checked it for the several first terms of expansion in the two-cut case). Then, we apply formulas for the third derivatives of the planar free energy and for the genus-one free energy obtained in the paper, and conjecture a diagrammatic interpretation of these formulas to be extended to higher genera.

1 Matrix models and generalized Whitham hierarchies

1.1 Matrix integrals and resolvents

Consider the 1MM integral[3]

$$\int_{N\times N} DX\, e^{-\frac{1}{\hbar}\mathrm{tr}\, V(X)} = e^{\mathcal{F}}, \tag{1}$$

where $V(X) = \sum_{n\geq 1} t_n X^n$, $\hbar = \frac{t_0}{N}$ is a formal expansion parameter, the integration goes over the $N\times N$ matrices, $DX \propto \prod_{ij} dX_{ij}$, and for generic potential one should consider the *holomorphic* version of (1) implying contour integration in complex plane in each variable. The topological expansion of the Feynman diagrams series is then equivalent to the expansion in even powers of $\hbar$ for

$$\mathcal{F} \equiv \mathcal{F}(\hbar, t_0, t_1, t_2, \ldots) = \sum_{h=0}^{\infty} \hbar^{2h-2}\mathcal{F}_h, \tag{2}$$

Customarily $t_0 = \hbar N$ is the scaled number of eigenvalues. We assume the potential $V(p)$ to be a polynomial of the fixed degree $m+1$, with the fixed constant "highest" time t_{m+1}.

The averages, corresponding to the partition function (1) are defined as usual:

$$\langle f(X)\rangle = \frac{1}{Z}\int_{N\times N} DX\, f(X)\, \exp\left(-\frac{1}{\hbar}\mathrm{tr}\, V(X)\right) \tag{3}$$

and it is convenient to use their generating functionals: the one-point resolvent

$$W(\lambda) = \hbar\sum_{k=0}^{\infty}\frac{\langle \mathrm{tr}\, X^k\rangle}{\lambda^{k+1}} \tag{4}$$

as well as the s-point resolvents ($s\geq 2$)

$$W(\lambda_1,\ldots,\lambda_s) = \hbar^{2-s}\sum_{k_1,\ldots,k_s=1}^{\infty}\frac{\langle \mathrm{tr}\, X^{k_1}\cdots \mathrm{tr}\, X^{k_s}\rangle_{\mathrm{conn}}}{\lambda_1^{k_1+1}\cdots\lambda_s^{k_s+1}} = \hbar^{2-s}\left\langle \mathrm{tr}\,\frac{1}{\lambda_1 - X}\cdots \mathrm{tr}\,\frac{1}{\lambda_s - X}\right\rangle_{\mathrm{conn}} \tag{5}$$

where the subscript "conn" pertains to the connected part.

These resolvents are obtained from the free energy $\mathcal{F}$ through the action

$$\begin{aligned} W(\lambda_1,\ldots,\lambda_s) &= \hbar^2\frac{\partial}{\partial V(\lambda_s)}\frac{\partial}{\partial V(\lambda_{s-1})}\cdots\frac{\partial\mathcal{F}}{\partial V(\lambda_1)} = \\ &= \frac{\partial}{\partial V(\lambda_s)}\frac{\partial}{\partial V(\lambda_{s-1})}\cdots\frac{\partial}{\partial V(\lambda_2)}W(\lambda_1), \end{aligned} \tag{6}$$

[3]There is a more consistent point of view on matrix integrals and loop equations below that implies introducing some background (polynomial) potential $V_0(x)$ into the exponential (1), i.e., shifting first several $t_k \to T_k + t_k$ and then looking at this matrix integral as a formal series in t_k's (see, e.g., [33, 20, 22] for a review). Then, the loop equations are iteratively solved. Such a framework is more effective when looking at the whole manifold of solutions to the loop equations. However, this is at the price of rigid fixing the number of cuts in the multicut solution from the very beginning. Since we will freely change the number of cuts (which is a smooth procedure specifically in the multicut solutions) and do not care of other than multicut solutions to the loop equations, we follow a less pure scheme (see, e.g., [4, 6, 34] and references therein) that basically would imply some re-summation of infinite series, etc. It allows us, however, instead of dealing with infinite series to deal with objects determined on Riemann surfaces.

of the loop insertion operator[4]

$$\frac{\partial}{\partial V(\lambda)} \equiv -\sum_{j=1}^{\infty} \frac{1}{\lambda^{j+1}} \frac{\partial}{\partial t_j}. \tag{7}$$

Therefore, if one knows exactly the one-point resolvent for arbitrary potential, all multi-point resolvents can be calculated by induction. In the above normalization, the genus expansion has the form

$$W(\lambda_1, \dots, \lambda_s) = \sum_{h=0}^{\infty} \hbar^{2h} W_h(\lambda_1, \dots, \lambda_s), \quad s \geq 1, \tag{8}$$

which is analogous to genus expansion (2).

The first in the chain of the loop equations [18] of the 1MM is [19]

$$\oint_{\mathcal{C}_{\mathcal{D}}} \frac{d\lambda}{2\pi i} \frac{V'(\lambda)}{x-\lambda} W(\lambda) \equiv \widehat{K} W(x) = W(x)^2 + \hbar^2 W(x,x) \tag{9}$$

where the linear integral operator $\widehat{K}$,

$$\widehat{K} f(x) \equiv \oint_{\mathcal{C}_{\mathcal{D}}} \frac{d\lambda}{2\pi i} \frac{V'(\lambda)}{x-\lambda} f(\lambda) = \left[V'(x) f(x)\right]_- \tag{10}$$

projects onto the negative powers[5] of λ. Hereafter, $\mathcal{C}_{\mathcal{D}}$ is a contour encircling all singular points of $W(\lambda)$, but not the point $\lambda = p$. Using Eq. (6), one can express the second term in the r.h.s. of loop equation (9) through $W(p)$, and Eq. (9) becomes an equation for the one-point resolvent (4).

Substituting the genus expansion (8) in Eq. (9), one finds that $W_h(\lambda)$ for $h \geq 1$ satisfy the equation

$$\left(\widehat{K} - 2W_0(\lambda)\right) W_h(\lambda) = \sum_{h'=1}^{h-1} W_{h'}(\lambda) W_{h-h'}(\lambda) + \frac{\partial}{\partial V(\lambda)} W_{h-1}(\lambda), \tag{11}$$

In Eq. (11), $W_h(\lambda)$ is expressed through only the $W_{h_i}(\lambda)$ for which $h_i < h$. This fact allows one to develop the iterative procedure.

To this end, one needs to use the asymptotics condition (which follows from the definition of the matrix integral)

$$W_h(\lambda)|_{\lambda \to \infty} = \frac{t_0}{\lambda} \delta_{h,0} + O(1/\lambda^2), \tag{12}$$

and manifestly solve (9) for genus zero. Then, one could iteratively find $W_h(\lambda)$ and then restore the corresponding contributions into the free energy by integration, since

$$W_h(\lambda) = \frac{\partial}{\partial V(\lambda)} \mathcal{F}_h, \quad h \geq 1. \tag{13}$$

1.2 Solution in genus zero

In genus zero, the loop equation (9) reduces to

$$\oint_{\mathcal{C}_{\mathcal{D}}} \frac{d\lambda}{2\pi i} \frac{V'(\lambda)}{x-\lambda} W_0(\lambda) = \left[V'(x) f(x)\right]_- = (W_0(x))^2 \tag{14}$$

[4] This operator contains all partial derivatives w.r.t. the variables t_k's. However, below we introduce additional variables S_i and, therefore, we use the partial derivative notation here.

[5] In order to prove it, one suffices to deform the integration contour to infinity to obtain

$$\oint_{\mathcal{C}_{\mathcal{D}}} \frac{d\lambda}{2\pi i} \frac{V'(\lambda)}{x-\lambda} f(\lambda) = V'(x) f(x) - \left[V'(x) f(x)\right]_+ = \left[V'(x) f(x)\right]_-$$

In order to solve this equation for the planar one-point resolvent $W_0(p)$, one suffices to note that

$$[V'(\lambda)W_0(\lambda)]_- = V'(\lambda)W_0(\lambda) - [V'(\lambda)W_0(\lambda)]_+ \tag{15}$$

and, due to (12), the last term in the r.h.s. is a polynomial of degree $m-1$, m being the degree of $V'(\lambda)$,

$$P_{m-1}(\lambda) = -[V'(\lambda)W_0(\lambda)]_+ = -\oint_{\mathcal{C}_\infty} \frac{dx}{2\pi i}\frac{V'(x)}{\lambda - x}W_0(x) \tag{16}$$

It can be also rewritten as action of the linear operator $\hat{r}_V(\lambda)$ acting to the free energy,

$$P_{m-1}(\lambda) = -\hat{r}_V(\lambda)\mathcal{F}_0, \qquad \hat{r}_V(\lambda) \equiv \sum_{k,l}(k+l+2)t_{k+l+2}\lambda^k \frac{\partial}{\partial t_l} \tag{17}$$

Then, the solution to (14) is

$$W_0(\lambda) = \frac{1}{2}V'(\lambda) - \frac{1}{2}\sqrt{V'(\lambda)^2 + 4P_{m-1}(\lambda)}, \tag{18}$$

where the minus sign is chosen in order to fulfill the asymptotics (12). For the polynomial potential of power $m+1$, the resolvent $W_0(\lambda)$ is a function on complex plane with m cuts, or on a hyperelliptic curve

$$y^2 = V'(\lambda)^2 + 4P_{m-1}(\lambda) \tag{19}$$

of genus $g = m-1$, conveniently presented introducing new variable y by

$$W_0(\lambda) = \frac{1}{2}\left(V'(\lambda) - y\right), \tag{20}$$

For generic potential $V(\lambda)$ with $m \to \infty$, curve (19) may have an infinite genus, but we can still consider solutions with only finite number n of cuts and separate the smooth part of curve (19) introducing

$$y \equiv M(\lambda)\tilde{y}, \quad \text{and “reduced” Riemann surface} \quad \tilde{y}^2 \equiv \prod\nolimits_{\alpha=1}^{2n}(\lambda - \mu_\alpha) \tag{21}$$

with all μ_α distinct. In what follows, we still assume $M(\lambda)$ to be a polynomial of degree $m-n$, keeping in mind that n is always finite and fixed, while $m \geq n$ can be chosen arbitrarily large. By convention, we set $\tilde{y}|_{\lambda\to\infty} \sim \lambda^n$, and $M(\lambda)$ is then[6]

$$M(\lambda) = \oint_{\mathcal{C}_\infty} \frac{dx}{2\pi i}\frac{V'(x)}{(x-\lambda)\tilde{y}(x)} \equiv \mathcal{M}\prod_i^{m-n}(\lambda - \lambda_i) \tag{23}$$

Note that the values of $M(\lambda)$ at branching points, $M_{m-n}(\mu_\alpha)$ coincide with the *first moments* $M_\alpha^{(1)}$ of the general matrix model potential (see [35, 34, 23]), while the higher moments are just derivatives at these points. Indeed, by their definition, these moments are

$$M_\alpha^{(p)} = \oint_{\mathcal{C}_\mathcal{D}} \frac{dx}{2\pi i}\frac{V'(x)}{(x-\mu_\alpha)^p\tilde{y}(x)}, \quad p \geq 1 \tag{24}$$

Then, one deforms the integration contour to infinity assuming $V'(x)$ is an entire function and, as in (23), make use of the asymptotic conditions (22), to replace $V'(x)$ in the numerator of (24) by $y(x) = M_{m-n}(x)\tilde{y}(x)$ subsequently evaluating the integral by taking the residue at the point $x = \mu_\alpha$ and obtaining

$$M_\alpha^{(p)} = \frac{1}{(p-1)!}\left.\left(\frac{\partial^{p-1}}{\partial\lambda^{p-1}}M(\lambda)\right)\right|_{\lambda=\mu_\alpha}, \quad \text{e.g., } M_\alpha^{(1)} = M(\mu_\alpha). \tag{25}$$

[6]Since, due to (19), at large λ

$$V'(\lambda) = y(\lambda) + \frac{2t_0}{\lambda} + O(1/\lambda^2) \tag{22}$$

Inserting the solution (21), (23) into (20) and deforming the contour, one obtains the planar one-point resolvent with an n-cut structure,

$$W_0(\lambda) = \frac{1}{2}\oint_{\mathcal{C}_\mathcal{D}} \frac{dx}{2\pi i}\frac{V'(x)}{\lambda - x}\frac{\tilde{y}(\lambda)}{\tilde{y}(x)}, \quad \lambda \notin \mathcal{D}. \tag{26}$$

The contour $\mathcal{C}_\mathcal{D}$ of integration here encircles the finite number n of disjoint intervals

$$\mathcal{D} \equiv \bigcup_{i=1}^{n} [\mu_{2i-1}, \mu_{2i}], \quad \mu_1 < \mu_2 < \ldots < \mu_{2n}. \tag{27}$$

Let us now discuss how many free parameters we have in our solution. If one does not keep genus of the curve fixed, it is given for a generic potential $V(\lambda)$ by $m+1$ times t_k, coefficients of $V'(\lambda)$, and $m-1$ coefficients p_k of the polynomial $P_{m-1}(\lambda)$, its leading coefficient being related to t_0. Indeed, using (12) and (18), one obtains

$$W_0(\lambda)|_{\lambda\to\infty} = -\frac{P_{m-1}(\lambda)}{V'(\lambda)} + \ldots = -\frac{p_{m-1}}{(m+1)t_{m+1}}\frac{1}{\lambda} + O(1/\lambda^2) = \frac{t_0}{\lambda} + O(1/\lambda^2) \tag{28}$$

Therefore, totally one has $2m+1$ parameters (including t_0). In fact, we usually fix the leading coefficient of the potential $V(\lambda)$ to be $1/m$ (see s.3), i.e. $\mathcal{M} = 1$ in (23), which leaves us with $2m$ parameters.

If, however, now one fixes the curve of genus $g = n-1$, (21), this imposes $m-n$ conditions of double points (coinciding branching cuts)[7]. Therefore, one then has $2m - (m-n) = m+n$ parameters. Of these, still $m+1$ parameters are t_k, which are variables in the loop equations, while $n-1$ parameters give the arbitrariness of solutions to these loop equations (possible different choices of the polynomial P_{m-1}).

One can arbitrary choose coordinates on this $n-1$-dimensional space of parameters. It turns out, however, that there is a distinguished set of n independent variables that parameterize solutions to the loop equations [6, 3],

$$S_i = \oint_{A_i} \frac{d\lambda}{4\pi i} y = \oint_{A_i} \frac{d\lambda}{4\pi i} M(\lambda)\tilde{y}, \tag{30}$$

where A_i, $i = 1, \ldots, n-1$ is the basis of A-cycles on the reduced hyperelliptic Riemann surface (21) (we may conveniently choose them to be the first $n-1$ cuts) see Fig.1.

Besides canonically conjugated A- and B-cycles, we also use the linear combination of B-cycles: $\bar{B}_i \equiv B_i - B_{i+1}$, $\bar{B}_{n-1} \equiv B_{n-1}$. Therefore, $\bar{B}$-cycles encircle the nearest ends of two neighbor cuts, while all B-cycles goes from a given right end of the cut to the last, nth cut. For the sake of definiteness, we order all points μ_α in accordance with their index so that μ_α is to the right of μ_β if $\alpha > \beta$.

1.3 Matrix eigenvalue picture: a detour

The variables S_i can be formally determined as eigenvalues of a differential operator in t_k's [22]. However, they can be also more "physically" interpreted in the quasi-classical picture of matrix eigenvalues (Coulomb gas) when their number (and, therefore, size of the matrix) goes to infinity. We come now to this interpretation.

[7] One can also give these conditions via different integral conditions. Say, one can get from (12) and (26), the asymptotic conditions

$$t_0\delta_{k,n} = \frac{1}{2}\oint_{\mathcal{C}_\mathcal{D}} \frac{d\lambda}{2\pi i}\frac{\lambda^k V'(\lambda)}{\tilde{y}}, \quad k = 0, \ldots, n. \tag{29}$$

These conditions are identically satisfied for $m = n$, since $V'(\lambda) \sim y(\lambda) + O(1/\lambda) = \tilde{y}(\lambda) + O(1/\lambda)$, while for $m > n$ they impose constraints.

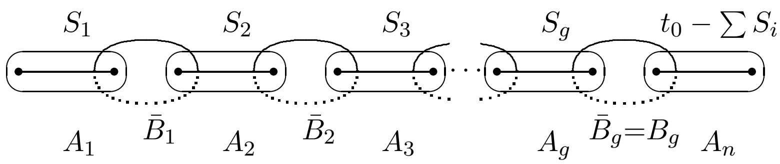


Figure 1: Structure of cuts and contours for the reduced Riemann surface.

To this end, let us first introduce the averaged eigenvalue distribution

$$\rho(\lambda) \equiv \frac{t_0}{N} \sum_i^N \langle \delta(\lambda - x_i) \rangle = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \Big(W(\lambda - i\epsilon) - W(\lambda + i\epsilon) \Big) \tag{31}$$

where x_i's are eigenvalues of the matrix X. In the planar limit, this quantity becomes

$$\rho_0(\lambda) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \Big(W_0(\lambda - i\epsilon) - W_0(\lambda + i\epsilon) \Big) = \frac{1}{2\pi} \operatorname{Im} y(\lambda) \tag{32}$$

and satisfies the equation[8]

$$\fint_{\mathcal{D}} \frac{\rho_0(\lambda)}{x - \lambda} d\lambda = \frac{1}{2} V'(x), \qquad \forall p \in \mathcal{D} \tag{33}$$

This averaged eigenvalue distribution becomes the distribution of eigenvalues in the limit when their number goes to infinity. For the illustrative purposes, let us do the matrix integral, (1) performing the matrix X in the form of $U \cdot \operatorname{diag}(x_i) \cdot U^{-1}$ with a unitary matrix U and then first making the integration over the "angular" variables U. Then, one comes to [36]

$$e^{\mathcal{F}} \sim \int \prod_i dx_i \prod_{i>j} (x_i - x_j)^2 e^{-\frac{1}{\hbar} \sum_i V(x_i)} =$$

$$= \int \prod_i dx_i e^{-\frac{1}{\hbar^2} \left(\int V(\lambda)\varrho(\lambda) - \int \varrho(\lambda)\varrho(\lambda') \log|\lambda - \lambda'| d\lambda d\lambda' \right)} \equiv \int \prod_i dx_i e^{\frac{1}{\hbar^2} S_{eff}} \tag{34}$$

where we introduced the eigenvalue distribution

$$\varrho(\lambda) \equiv \frac{t_0}{N} \sum_i \delta(\lambda - \lambda_i) \tag{35}$$

Now, in the limit of large N, one can use the saddle point approximation to obtain the equation for $\varrho(\lambda)$. However, one also need to take into account the constraint

$$\int \varrho(\lambda) d\lambda = t_0 \tag{36}$$

by adding to S_{eff}, (34) the Lagrange multiplier term $\Pi_0(\int \varrho - t_0)$. Note also that $\varrho(\lambda)$ is a non-negative density. This finally leads to the saddle point equation

$$2 \int \varrho(\lambda) \log|x - \lambda| d\lambda = V(x) + \Pi_0, \qquad \forall x \in \text{support of } \varrho \tag{37}$$

The derivative of this equation w.r.t. p coincides with (33).

[8] Indeed, by definition

$$\oint_\infty \frac{W_0(\lambda)}{x - \lambda} d\lambda = 0$$

Now, using (20) and the definition (31) and pulling out the contour from infinity, one easily comes to this equation.

The variable t_0 plays role of the (normalized) total number of eigenvalues,

$$t_0 = \frac{1}{4\pi i}\oint_{\mathcal{C}_{\mathcal{D}}} y d\lambda = -\frac{1}{2}\mathrm{res}_\infty(y d\lambda) \tag{38}$$

and the support of ϱ is $\mathcal{D}$ that consists of N segments $\mathcal{D}_i$. Now, following [6, 3], one may fix the (occupation) numbers of eigenvalues in each of the segments, S_i (30), $i = 1, \ldots, n-1$. We assume the occupation number for the last, nth cut to be $t_0 - \sum_{i=1}^{n-1} S_i \equiv S_n$. [9] (Obviously, no new parameters S_i arise in the one-cut case.) We formally attain this by introducing the corresponding chemical potentials (Lagrange multipliers) Π_i, $i = 1, \ldots, n-1$, in the variational problem for the free energy, which therefore becomes in the planar limit

$$\begin{aligned} S_{eff}\left[\varrho; S_i, t_0, t_k\right] &= -\int_{\mathcal{D}} V(\lambda)\varrho(\lambda)d\lambda + \iint_{\mathcal{D}} \varrho(\lambda)\log|\lambda-\lambda'|\,\varrho(\lambda')d\lambda d\lambda' + \\ &\quad -\Pi_0\left(\int_{\mathcal{D}}\varrho(\lambda)d\lambda - t_0\right) - \sum_{i=1}^{n-1}\Pi_i\left(\int_{D_i}\varrho(\lambda)d\lambda - S_i\right). \end{aligned} \tag{39}$$

while the saddle point equation becomes

$$2\int \varrho(\lambda)\log|x-\lambda|d\lambda = V(x) + \Pi_i + \Pi_0, \qquad \forall x \in \mathcal{D}_i \tag{40}$$

and its derivative still coincides with (33).

Therefore, with generic values of the constants Π_i, $\varrho_c(\lambda)$ gives the general solution to (33) (or the planar limit of the loop equation): these constants describe the freedom one has when solving the loop equation[10]. However, in the matrix model integral (where there are no any chemical potentials) one would further vary $\mathcal{F}_0$ w.r.t. Π_i to find the "true" minimum of the eigenvalue configuration,

$$\frac{\partial \mathcal{F}_0}{\partial S_i} = 0, \qquad \forall i \tag{41}$$

This is a set of equation that fixes concrete values of S_i and Π_i in the matrix integral.

Let us now calculate the derivative of $\mathcal{F}_0$ (39) w.r.t. S_i. From (39), one has

$$\left.\frac{\partial S_{eff}}{\partial S_i}\right|_{\varrho=\rho_0} = -\int_{\mathcal{D}} d\lambda \frac{\partial \rho_0(\lambda)}{\partial S_i}\left(V(\lambda) - 2\int_{\mathcal{D}} d\lambda' \log(\lambda-\lambda')\rho_0(\lambda')\right) \tag{42}$$

The expression in the brackets on the r.h.s. of (42) is almost a variation of (39) w.r.t. the eigenvalue density, which is

$$\begin{aligned} 0 = \left.\frac{\delta S_{eff}}{\delta \rho(\lambda)}\right|_{\varrho=\rho_0} &= V(\lambda) - 2\int_{\mathcal{D}} d\lambda' \log(\lambda-\lambda')\rho_0(\lambda') + \Pi_i + \Pi_0 \\ &\quad \text{for } \lambda \in D_i \subset \mathcal{D}. \end{aligned} \tag{43}$$

It is therefore a step function, $h(\lambda)$ which is constant equal to $\zeta_i \equiv -\Pi_0 - \Pi_i$ on each cut A_i. One then has

$$\begin{aligned} \frac{\partial \mathcal{F}_0}{\partial S_i} = \left.\frac{\partial S_{eff}[\varrho]}{\partial S_i}\right|_{\varrho=\rho_0} &= -\int_{\mathcal{D}} d\lambda \frac{\partial \rho_0(\lambda)}{\partial S_i} h(\lambda) = -\frac{1}{4\pi i}\sum_{j=1}^{n}\zeta_j \frac{\partial}{\partial S_i}\oint_{A_j} y(\lambda)d\lambda = \\ &= -\sum_{j=1}^{n}\zeta_j\frac{\partial S_j}{\partial S_i} = -\zeta_i + \zeta_n = \Pi_i. \end{aligned} \tag{44}$$

[9] It is sometimes convenient to consider S_n instead of t_0 as a canonical variable. However, in all instants we use S_n, we specially indicate it for not confusing S_n with the "genuine" filling fraction variables S_i, $i = 1, \ldots, n-1$.

[10] Note that, instead of fixing the occupation numbers, one could use other ways to fix a solution to the loop equations, see [20, 21, 22].

In particular,

$$\frac{\partial \mathcal{F}_0}{\partial t_0} = \Pi_0 \tag{45}$$

In [4] it was proved that the difference of values of Π_i on two neighbour cuts is equal to [11]

$$\zeta_{i+1} - \zeta_i = 2\int_{\mu_{2i}}^{\mu_{2i+1}} W_0(\lambda)d\lambda \tag{46}$$

i.e.

$$\Pi_i = (\zeta_{i+1}-\zeta_i)+(\zeta_{i+2}-\zeta_{i+1})+\ldots+(\zeta_{n-1}-\zeta_{n-2})+(\zeta_n-\zeta_{n-1}) = \oint_{\bar{B}_i\cup\bar{B}_{i+1}\cup\ldots\cup\bar{B}_g} yd\lambda = \oint_{B_i} yd\lambda \tag{47}$$

Note that one can calculate the planar limit free energy that can be obtained via substituting the saddle point solution ϱ into (39) is

$$\mathcal{F}_0 = S_{eff}\left[\varrho_c\right] = -\frac{1}{2}\int_{\mathcal{D}} V(\lambda)\varrho_c(\lambda)d\lambda + \frac{1}{2}\Pi_0 t_0 + \frac{1}{2}\sum_{i=1}^{n-1}\Pi_i S_i \tag{48}$$

In the paper, we choose the solution to the loop equation with fixed occupation numbers, (30). Note that fixing the chemical potentials (44)-(47) instead, (41), corresponds just to interchanging A- and B-cycles on the Riemann surface (19). However, $\mathcal{F}_0$ is *not* modular invariant. Under the change of homology basis, $\mathcal{F}_0$ transforms in accordance with the duality transformations [37] (which is a particular case of behaviour of $\mathcal{F}_0$ under the general transformations, [38]). The higher-genus corrections become also basis-dependent: choosing S_i or Π_i as independent variables, one obtains different expressions, say, for the genus-one free energy, see s.4.3.

In the next two subsections we are going to demonstrate that the planar loop equation solution with fixed occupation numbers corresponds to a Seiberg-Witten-Whitham system.

1.4 Seiberg–Witten-Whitham theory

Seiberg-Witten system. We call the SW system [7][12] the following set of data:

- a family $\mathcal{M}$ of Riemann surfaces (complex curves) $\mathcal{C}$ so that the dimension of moduli space of this family coincides with the genus[13];
- a meromorphic differential dS whose variations w.r.t. moduli of curves are holomorphic (this implies existence of a connection on moduli space, so that this statement has a strict sense, see e.g. [12, 39]).

These data allow to define a SW *prepotential* [7] related to an integrable system [9, 40, 41, 12, 14]. First, one introduces the variables (whose number coincides with the genus of $\mathcal{C}$)

$$S_i \equiv \frac{1}{4\pi i}\oint_{A_i} dS \tag{49}$$

where A_i are A-cycles on $\mathcal{C}$. As soon as $\frac{\partial dS}{\partial S_i}$ is holomorphic, from the definition (49) of S_i and the obvious relation $\partial S_j/\partial S_i = \delta_{ji}$, one finds that

$$\frac{1}{4\pi i}\oint_{A_j}\frac{\partial dS}{\partial S_i} = \delta_{ji}, \tag{50}$$

[11]The simplest way to prove it is to define function $h(\lambda)$ outside the cuts: $h(\lambda) = V(\lambda) - 2\int_{\mathcal{D}} d\lambda' \log(\lambda-\lambda')\rho_0(\lambda')$ and note that $h'(\lambda)|_{\lambda\notin\mathcal{D}} = 2W_0(\lambda)$.

[12]Various properties of such systems can be found in [12, 13, 14].

[13]More generally, this can be extended to the curves with auxiliary involution, or certain directions in moduli space can be "frozen" in a different way, certain examples can be found in [28].

i.e.,

$$\frac{\partial dS}{\partial S_i} = d\omega_i \tag{51}$$

where $d\omega_i$ are the canonically normalized holomorphic 1-differentials,

$$\frac{1}{4\pi i}\oint_{A_i} d\omega_j = \delta_{ij} \tag{52}$$

Introducing B-cycles conjugated to A-cycles: $A_i \circ B_j = \delta_{ij}$, where $\circ$ means intersection form, one obtains that

$$\frac{\partial}{\partial S_i}\oint_{B_j} dS = \oint_{B_i} d\omega_j = T_{ij} \tag{53}$$

is the period matrix of $\mathcal{C}$ and is therefore symmetric[14]. Hence, there exists a locally defined function F such that

$$\frac{\partial F}{\partial S_i} = \oint_{B_i} dS. \tag{55}$$

and this function is called a prepotential.

Generalized Whitham system. We now extend the Seiberg-Witten system to the quasiclassical or generalized Whitham system [9] by introducing extra parameters or times t_k into the game, we do it mostly following [10, 11][15]. In order to construct a Whitham system, one needs to add to the SW data a set of jets of local coordinates in the vicinity of punctures on $\mathcal{C}$. In particular case of (19), (21) and these points (the singularities of $yd\lambda$ coincide with two λ-infinities, where we choose the local parameter $\xi = \frac{1}{\lambda}$. We then introduce a set of *meromorphic* differentials $d\Omega_k$ with the poles only at these punctures (as the hyperelliptic curve (19), (21) is invariant w.r.t. the involution $y \to -y$, from now on we just work with either of the two infinities, see [10, 11]) with the behavior

$$d\Omega_k = \pm\frac{k}{2}\left(\xi^{-k-1} + O(1)\right)d\xi, \text{ for } \xi \to 0, \ k > 0 \tag{56}$$

where signs are different for the different infinities. We also introduce the bipole or 3rd-kind Abelian differential, with two simple poles, located at two infinities:

$$\mathrm{res}_{\infty} d\Omega_0 = -\mathrm{res}_{\infty_-} d\Omega_0 = -1 \tag{57}$$

Then, the generalized Whitham system is generated by a set of equations on these differentials and on the holomorphic differentials $d\omega_i$:

$$\frac{\partial d\Omega_p}{\partial t_k} = \frac{\partial d\Omega_k}{\partial t_p}, \qquad 2\frac{\partial d\Omega_k}{\partial S_i} = \frac{\partial d\omega_i}{\partial t_k}, \qquad \frac{\partial d\omega_i}{\partial S_j} = \frac{\partial d\omega_j}{\partial S_i} \tag{58}$$

where the partial derivatives are supposed to be taken at constant hyperelliptic co-ordinate λ.

These equations imply that there exists a differential dS such that (see also (51)

$$\frac{\partial dS}{\partial S_i} = d\omega_i, \qquad \frac{\partial dS}{\partial t_k} = 2d\Omega_k \tag{59}$$

[14]This follows from the Riemann bilinear relations for canonical holomorphic differentials (51)

$$\begin{aligned} 0 = \int_{\Sigma_g} d\omega_i \wedge d\omega_j &= \sum_k \left(\oint_{A_k} d\omega_i \oint_{B_k} d\omega_j - \oint_{A_k} d\omega_j \oint_{B_k} d\omega_i\right) = \\ &= 4\pi i (T_{ij} - T_{ji}) \end{aligned} \tag{54}$$

[15]One can introduce these extra parameters in the context of supersymmetric SW gauge theories, even without reference to an integrable system *a priori*, see [42] and references therein.

Note, however, that the meromorphic differentials (56), (57) are defined up to linear combinations of the holomorphic differentials. Since we consider S_i and t_k to be independent variables, this ambiguity is removed merely by imposing the condition [10, 11]

$$\frac{\partial S_i}{\partial t_k} = \frac{1}{2\pi i}\oint_{A_i} d\Omega_k = 0 \ \forall\ i,k \tag{60}$$

Now we can invariantly introduce variables t_k via the relations[16]

$$t_k = -\frac{1}{k}\mathrm{res}_\infty\left(\lambda^{-k}dS\right), \quad k = 1,\dots,m, \quad t_0 = \mathrm{res}_\infty dS \ \ (\text{cf. } (38)) \tag{61}$$

Then, one defines the prepotential that depends on both S_i and t_k via the old relation (55) and the similar relations

$$\frac{\partial F}{\partial t_k} = \frac{1}{2}\mathrm{res}_{\lambda=\infty}(\lambda^k dS) \equiv \frac{1}{2}v_k, \quad k = 1,\dots,m. \tag{62}$$

$$\frac{\partial F}{\partial t_0} = \int_{\infty_-}^{\infty_+} dS \tag{63}$$

The latter integral, which is naively divergent, is still to be supplemented with some proper regularization, we discuss this in detail in the next subsection.

In fact, one still needs to prove such a prepotential exists [10, 11], by checking the second derivatives are symmetric. This can be verified using the Riemann bilinear relations,

$$\begin{aligned}
\frac{\partial v_k}{\partial S_i} &= 2\mathrm{res}_\infty\left(\Omega_k d\omega_i\right) = \oint_{\partial\Sigma_g}\Omega_k d\omega_i = \\
&= 2\sum_{l=1}^{g}\left(\int_{B_l}\Omega_k^+ d\omega_i - \int_{B_l}\Omega_k^- d\omega_i\right) - 2\sum_{l=1}^{g}\left(\int_{A_l}\Omega_k^+ d\omega_i - \int_{A_l}\Omega_k^- d\omega_i\right) = \\
&= 2\sum_{l=1}^{g}\left(\oint_{A_l} d\Omega_k\oint_{B_l} d\omega_i - \oint_{B_l} d\Omega_k\oint_{A_l} d\omega_i\right) = 2\oint_{B_i} d\Omega_k = \frac{\partial \Pi_i}{\partial t_k}.
\end{aligned} \tag{64}$$

Here $\partial\Sigma_g$ is the cut reduced Riemann surface (21) (see fig. 2), $\Omega_k^\pm$ are values of $\Omega_k \equiv \int d\Omega_k$ on two sides of the corresponding cycle (they are related by the integral over the dual cycle), and we have used (72).

Similarly, for the derivatives w.r.t. the times t_k, one has analogously to (64)

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial v_k}{\partial t_p} - \frac{\partial v_p}{\partial t_k}\right) &= \mathrm{res}_\infty\left((\Omega_k)_+ d\Omega_p - d\Omega_k(\Omega_p)_+\right) = \\
&= \mathrm{res}_\infty\left(\Omega_k d\Omega_p\right) = \oint_{\partial\Sigma_g}\Omega_k d\Omega_p = \\
&= \sum_{l=1}^{g}\left(\oint_{A_l} d\Omega_k\oint_{B_l} d\Omega_p - \oint_{B_l} d\Omega_k\oint_{A_l} d\Omega_p\right) = 0.
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
0 &= \int_{\Sigma_g} d\omega_i\wedge d\Omega_0 = \sum_{l=1}^{g}\left(\oint_{A_l} d\omega_i\oint_{B_l} d\Omega_0 - \oint_{A_l} d\Omega_0\oint_{B_l} d\omega_i\right) + \\
&\quad +\mathrm{res}_\infty(d\omega_i)\int_{\infty_-}^{\infty_+} d\Omega_0 - \mathrm{res}_\infty(d\Omega_0)\int_{\infty_-}^{\infty_+} d\omega_i = \\
&= \oint_{B_i} d\Omega_0 - \int_{\infty_-}^{\infty_+} d\omega_i.
\end{aligned} \tag{66}$$

[16] In what follows, we call the ∞-point the point ∞_+, or $\lambda = \infty$ on the "upper" (physical) sheet of hyperelliptic Riemann surface (21) corresponding to the positive sign of the square root.

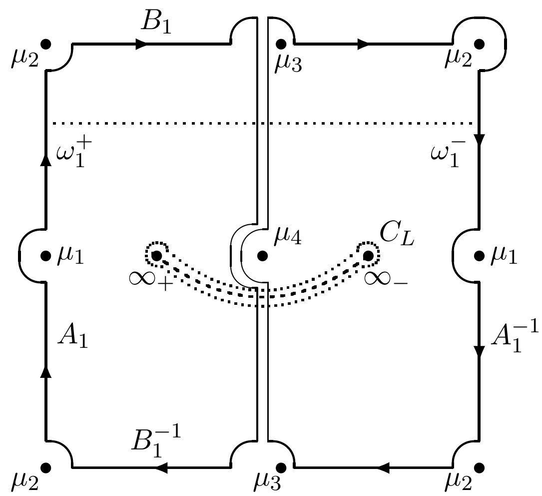


Figure 2: Cut Riemann surface from Fig. 3. The integral over the boundary can be divided into several pieces (see formula (118)). In the process of computation we use the fact that the boundary values of Abelian integrals $\omega_j^{\pm}$ on two copies of the cut differ by period integral of the corresponding differential $d\omega_j$ over the dual cycle. We add the two infinity points and the additional, nth, cut dividing the surface in two sheets. We present the logarithmic cut between these two points (about which we draw the standard integration contour C_L). Integrals over small circles around the points μ_α are relevant only when calculating the third-order derivatives w.r.t. the canonical variables t_I.

1.5 Free energy as prepotential of SWW system

We now associate a Seiberg-Witten-Whitham (SWW) system with the planar limit of the matrix-model free energy.

Matrix integral as a SW system. The family $\mathcal{M}$ in this case is the family of $h = n-1$ reduced Riemann surfaces described by (19) or (21). In different words, there is no information in $\mathcal{M}$ about the additional polynomial $M_{m-n}(\lambda)$, which is present, however, for (21) in the differential dS. The role SW differential is played by

$$dS = y d\lambda \tag{67}$$

Consider its variation w.r.t. S_i, the variation over moduli $\mathcal{M}$ does not change the genus of the reduced Riemann surface as well as the highest degree of the additional polynomial $M_{m-n}(\lambda)$. Moreover, considering the times of the potential $V'_m(\lambda)$ to be *independent* on the parameters S_i, we assume $\delta V'/\delta S_i \equiv 0$. Below, by δ and δ_S we denote the respective general variation and variation specifically w.r.t. the moduli parameters S_i.

Using (19), (21), one obtains for the *general* variation δdS: [17]

$$\begin{aligned} \delta dS &= \delta\left(M_{m-n}(\lambda)\tilde{y}(\lambda)\right) d\lambda = \\ &= \frac{g_{2n}(\lambda)\delta M_{m-n}(\lambda) + \frac{1}{2}M_{m-n}(\lambda)\delta g_{2n}(\lambda)}{\tilde{y}(\lambda)} d\lambda, \end{aligned} \tag{68}$$

where the polynomial expression in the numerator is of maximum degree $m+n-1$ (since the highest term of M_{m-n} is fixed). On the other hand, under δ_S which does not alter the potential, we obtain from (19), (21) that

$$\delta_S dS = -\frac{1}{2}\frac{\delta_S P_{m-1}(\lambda)}{M_{m-n}(\lambda)\tilde{y}(\lambda)} d\lambda. \tag{69}$$

[17] Note that the variation δ differs nevertheless from loop insertion (7) because the former does not change, by definition, the degree of the polynomial $M_{m-n}(\lambda)$.

Because this variation is just a particular case of (68), we obtain that zeros of $M_{m-n}(\lambda)$ in the denominator of (69) must be exactly *cancelled* by zeros of the polynomial $\delta_S P_{m-1}(\lambda)$ in the numerator, so the maximum degree of the polynomial in the numerator is $n-2$ (because, again, the highest-order term of $P_{m-1}(\lambda)$ is fixed by asymptotic condition (22) and is not altered by variations δ_S). We then come to the crucial observation that the variation $\delta_S dS$ is *holomorphic* on the curve (21), as it should be for the SW differential.

The canonical 1-differentials on the reduced Riemann surface $\tilde{y}(\lambda)$ have the form

$$\frac{\partial dS}{\partial S_i} = d\omega_i = \frac{H_i(\lambda)d\lambda}{\tilde{y}(\lambda)} \tag{70}$$

and $H_i(\lambda)$ are polynomials of degrees at most $n-2$. The normalization condition (52) unambiguously fixes the form of the polynomials $H_i(\lambda)$.

Comparing now (47) and (55), one could indeed identify the planar limit $\mathcal{F}_0$ of the 1MM free energy with an SW prepotential F. However, in order to specify this equivalence further, one needs to work out the t-dependence of the free energy, i.e. consider the generalized Whitham system.

Matrix integral as a Whitham system. To this end, let us check that differential (67) on curve (21) with the relation for moduli (19), (21) does satisfy (59).

Indeed, we have proved the first set of relations (59) in the previous paragraph. Now let us consider variations of the potential, i.e., variations w.r.t. Whitham times t_k. Then, we obtain instead of (69)

$$\delta dS = -\frac{1}{2}\frac{\delta\left((V'_m)^2(\lambda) - P_{m-1}(\lambda)\right)}{M_{m-n}(\lambda)\tilde{y}(\lambda)}d\lambda \tag{71}$$

while (68) still holds. Repeating the argument of the previous paragraph, we conclude that the zeroes of $M_{n-k}(\lambda)$ cancel from the denominator and, therefore, the variation may have pole only at $\lambda=\infty$, or $\eta=0$, i.e., at the puncture. In order to find this pole, we use (68), which implies that $dS = M_{m-n}(\lambda)\tilde{y}(\lambda)d\lambda \to (V'_m(\lambda)+O(\frac{1}{\lambda}))d\lambda$ and, therefore, the variation of dS at large λ is completely determined by the variation of $V'_m(\lambda)$. Parameterizing $V(\lambda) = \sum_{k=1}^{m+1} t_k\lambda^k$, we obtain (59) up to a linear combination of holomorphic differentials. One may fix the normalization of $d\Omega_k$ that are also defined up to a linear combination of holomorphic differentials in order to make Eq. (59) *exact*. We already discussed (see 60) that this normalization and, therefore, unambiguous way the variables S_i depend on the coefficients of P_{m-1} is fixed by the condition [10, 11]

$$\frac{\partial S_i}{\partial t_k} = \frac{1}{2\pi i}\oint_{A_i} d\Omega_k = 0 \ \forall\, i,k, \quad \text{or} \quad \frac{\partial S_i}{\partial V(\lambda)} = 0 \ \ i=1,\dots,n-1 \tag{72}$$

The derivatives of dS w.r.t. the times are

$$\begin{aligned} 2d\Omega_k \equiv \frac{\partial dS}{\partial t_k} &= \frac{V'(\lambda)k\lambda^{k-1}d\lambda}{y} + \frac{1}{2}\sum_{j=0}^{m-2}\frac{\partial P_j}{\partial t_k}\frac{\lambda^j d\lambda}{y} \\ &\equiv \frac{H_{n+k-1}(\lambda)d\lambda}{\tilde{y}(\lambda)}, \end{aligned} \tag{73}$$

and the normalization conditions (72) together with the asymptotic expansion

$$\begin{aligned} 2d\Omega_k(\lambda)|_{\lambda\to\infty} &= k\lambda^{k-1}d\lambda + O(\lambda^{-2})d\lambda = \\ &= k\lambda^{k-1}d\lambda + \sum_{m=1}^{\infty} c_{km}\lambda^{-1-m}d\lambda. \end{aligned} \tag{74}$$

fixes uniquely the coefficients of the corresponding polynomials H_{n+k-1} of degrees $n+k-1$.

Now we again find that the prepotential F defined in (62) *coincides* with $\mathcal{F}_0$. To this end, we apply the formula similar to (42) with the only difference that the potential $V(\lambda)$ itself is changed. We then obtain (see (42)-(44))

$$
\begin{aligned}
\frac{\partial \mathcal{F}_0}{\partial t_k} &= -\frac{1}{4\pi i}\oint_{\mathcal{D}} d\lambda \frac{\partial y(\lambda)}{\partial t_k}\cdot h(\lambda) - \frac{1}{4\pi i}\oint_{\mathcal{D}} d\lambda y(\lambda)\lambda^k = \\
&= -\sum_{i=1}^{n-1}\frac{\partial S_i}{\partial t_k}(\zeta_i - \zeta_n) + \frac{1}{2}\mathrm{res}_{\lambda=\infty}\lambda^k dS,
\end{aligned}
\tag{75}
$$

which by virtue of (72) gives (62).

On t_0-dependence of prepotential. Thus, we have proved the derivatives of the SWW prepotential and of the matrix model free energy w.r.t. S_i's and t_k's coincide. However, there is a subtlety of exact definition of integral (63) for $\partial F/\partial t_0$. This quantity is to be compared with the derivative $\partial\mathcal{F}_0/\partial t_0$, (45) equal to the integral $\oint_{\mathcal{D}} \log|\lambda - y| dS - V(y)$, where the reference point y is to be chosen on the last, nth, cut, while the expression itself does not depend on the actual local position of the reference point. It is convenient to choose it to be $\mu_{2n} \equiv b_n$—the rightmost point of the cut. We can then invert the contour integration over the support $\mathcal{D}$ to the integral along the contour that runs first along the upper side of the logarithmic cut from b_n to a regularization point Λ, then over the circle C_Λ of large radius $|\Lambda|$ and then back over the lower side of the logarithmic cut in the complex plane. In order to close the contour on the hyperelliptic Riemann surface under consideration, we must add the integration over the corresponding contour on the *second* sheet of the surface as shown in Fig. 3; we let C_L denote the completed integration contour, and it is easy to see that such an additional integration just double the value of the integral.

It is easy to see that all the singularities appearing at the upper integration limit (i.e., at the point Λ) are exactly cancelled by the contribution coming when integrating the expression $dS\log(\lambda - b_n)$ along the circle C_Λ; in fact, the latter can be easily done, the result is $-2\pi i(S(\Lambda) - \{S(b_n)\}_+)$, where the function $S(\lambda)$ is the (formal) primitive of dS (which includes the logarithmic term), and the symbol $\{\cdot\}_+$ denotes the projection to the strictly polynomial part of the expression in the brackets. Using the large-λ asymptotic expansion of the differential dS,

$$
dS(\lambda)|_{\lambda\to\infty} = V'(\lambda)d\lambda + \frac{t_0}{\lambda}d\lambda + O(\lambda^{-2})d\lambda,
\tag{76}
$$

we obtain that $(S(b_n))_+$ just cancels the term $V(b_n)$, and we eventually find that

$$
\begin{aligned}
\frac{\partial F}{\partial t_0} &= 1/2\left(\oint_{C_L}\log(\lambda - b_n)dS - 2V(b_n)\right) = \\
&= 2\pi i\left(\int_{b_n}^{\Lambda} dS - S(\Lambda)\right),
\end{aligned}
\tag{77}
$$

where C_L is the contour described above (see also Fig. 3), which by convention encircles the logarithmic cut between two infinities on two sheets of the Riemann surface and passes through the last, nth, cut.

Thus, we proved that $\partial F/\partial t_0$ coincides $\partial\mathcal{F}_0/\partial t_0$, (45). This completes the proof that the derivatives of F and $\mathcal{F}_0$ w.r.t. all S_i's and t_k, $k = 0, 1, ...$ coincide. Therefore, the planar limit $\mathcal{F}_0$ of the 1MM free energy is indeed the SWW prepotential or quasiclassical tau-function. Note that this identification of the matrix model free energy and the SWW prepotential is crucially based on formula (72) which fixes solutions to the loop equations (see ss.1.2-1.3). Moreover, making higher genera calculations, we shall solve the loop equations with similar additional constraints that fix the solution, see s.4 and formula (137) below.

We now introduce the (complete) set of canonical variables $\{S_i,\ i = 1, \dots, n-1;\ t_0;\ t_k,\ k = 1, \dots, m\}$, which we uniformly denote $\{t_I\}$ (in what follows, Latin capitals indicate any quantity from

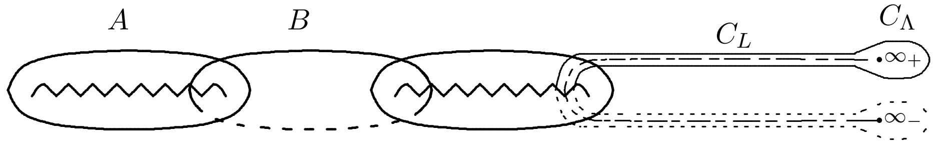


Figure 3: Cuts in the λ-, or "eigenvalue," plane for the planar limit of 1MM. The eigenvalues are supposed to be located "on" the cuts. The distribution of eigenvalues is governed by the period integrals $S_i = \oint_{A_i} \rho(\lambda) d\lambda$ along the corresponding cycles, and the dependence of the free energy on "occupation numbers" S_i is given by quasiclassical tau-function $\frac{\partial \mathcal{F}_l}{\partial S_i} = \oint_{B_i} y(\lambda) d\lambda$. We must add the logarithmic cut between two copies of the infinity on two sheets of the hyperelliptic Riemann surface in order to calculate the derivative w.r.t. the variable t_0.

this set). From (51), (57), and (73), we then obtain the general relation

$$\frac{\partial dS}{\partial t_I} \equiv d\Omega_I = \frac{H_I(\lambda) d\lambda}{\tilde{y}(\lambda)}, \tag{78}$$

where $H_I(\lambda)$ are polynomials.

Asymptotic formulas (57) and (74) imply that *derivatives* of all the quantities $d\Omega_I$ w.r.t. any parameter are regular at infinity and may have singularities *only* at the ramification points μ_α of reduced Riemann surface (21).

2 Second derivatives of free energy

2.1 Bergmann bidifferential

In the previous section we demonstrated how to calculate one-point resolvent. This required the knowledge of the first derivatives of the matrix model free energy. These could be expressed through the local quantities (integrals of differentials) on Riemann surfaces (and allowed us to associate our construction with the Whitham system).

In this section, we apply this procedure to the two-point resolvent, which requires the knowledge of the second derivatives. Here, instead of differentials with some prescribed properties of holomorphicity, the main object we need is a *bi*differential: the *Bergmann kernel* (canonically normalized bidifferential in Fay's terminology) which is the bi-differential on a Riemann surface Σ_g being the double derivative of logarithm of the prime form $E(P,Q)$ such that it is symmetrical in its arguments $P, Q \in \Sigma_g$ and has the only singularity at the coinciding arguments where it has the behavior (see [43])

$$B(P,Q) = \left(\frac{1}{(\xi(P) - \xi(Q))^2} + \frac{1}{6} S_B(P) + o(1) \right) d\xi(P) d\xi(Q), \tag{79}$$

in some local coordinate $\xi(P)$ in the vicinity of a point $P \in \mathcal{C}$; $S_B(P)$ is the Bergmann projective connection ($S_B(P)$ transforms as a quadratic differential plus the Schwarzian derivative under an arbitrary variable changing; this transformation law is the same as for the energy–momentum tensor of the (free) scalar field, see [44]). As it stands, we can add to (79) any bilinear combination of Abelian 1-differentials $d\omega_i$; we fix the normalization claiming vanishing all the integrals over A-cycles of $B(P,Q)$:

$$\oint_{A_i} B(P,Q) = 0, \text{ for } i = 1, \dots, g, \tag{80}$$

and, due to the symmetricity property, the integral may be taken over any of the variables P or Q.

The prime form, a fundamental object on the Riemann surface is defined as follows. Consider the Jacobian J, which is a h-dimensional torus defined by the period map of the curve Σ_g. Recall that the Abel map $\Sigma_g \mapsto J: \ P \rightarrow \vec{x}(P) \equiv \left\{\int_{P_0}^P d\omega_i\right\}$, where P_0 is a reference point, set into the correspondence to each point P of the complex curve the vector in the Jacobian, and we also introduce the theta function $\Theta_{[\alpha]}(\vec{x})$ of an odd characteristic $[\alpha]$ that becomes zero at $\vec{x}=0$. Introduce also the "normalizing" holomorphic half-differentials $h_\alpha(P)$ determined for the points of $P \in \Sigma_g$ and characteristics α by

$$h_\alpha^2(P) = \sum_{i=1}^{g} \frac{\partial \Theta_{[\alpha]}(0)}{\partial x_i} d\omega_i(P).$$

The explicit expression for the prime form $E(P,Q)$ that has a single zero on the Riemann surface Σ_g then reads

$$E(P,Q) = \frac{\Theta_{[\alpha]}(\vec{x}(P)-\vec{x}(Q))}{h_\alpha(P)h_\alpha(Q)} \tag{81}$$

while the Bergmann kernel is just

$$B(P,Q) = d_P d_Q \log E(P,Q) \tag{82}$$

which can be immediately stated by analytical properties and zero A-periods.

Note that, up to a holomorphic part, the Bergmann bidifferential is nothing but the scalar Green function, see [10, 45, 46]. Another useful Green function, that is, the fermionic one, $\Psi_e(P,Q)$ is defined to be a holomorphic 1/2-differential in both variables but the point $P=Q$ where it has the first order pole with unit residue. It also depends on the choice of theta-characteristics e (boundary conditions for the fermions) and is manifestly given by

$$\Psi_e(P,Q) = \frac{\Theta_e(\vec{x}(P)-\vec{x}(Q))}{\Theta_e(\vec{0})E(P,Q)} \tag{83}$$

The square of the Szegö kernels and the bi-differential $B(P,Q)$ are related by the identity [43] (Proposition 2.12; see also Appendices A,B in [10, 12]):

$$\Psi_e(P,Q)\Psi_{-e}(P,Q) = B(P,Q) + d\omega_i(P)d\omega_j(Q)\frac{\partial^2}{\partial x_i \partial x_j}\log \Theta_e(\vec{x})\Bigg|_{\vec{x}=0} \tag{84}$$

This allows one to express $B(P,Q)$ through the square of the Szegö kernel (note that, for the half-integer characteristics, $-e$ is equivalent to e).

As we shall see in the next subsection, the Bergmann kernel generates the differentials $d\Omega_k$. Therefore, formula (82) would allow one to express these latter through the prime form. Similarly, the bipole differential (57) can be rewritten through the prime form as

$$d\Omega_0 = d\log\frac{E(P,\infty_+)}{E(P,\infty_-)} \tag{85}$$

The primitive of differential (85) (which we need in what follows) then obviously develops the logarithmic cut between the points of two infinities on the Riemann surface.

2.2 2-point resolvent

Now one can easily express the 2-point resolvent $W_0(\lambda,\mu)$ in terms of $B(P,Q)$ on hyperelliptic curve (21), where we now use the hyperelliptic co-ordinate $\lambda = \xi(P)$ and $\mu = \xi(Q)$. Indeed, let us use (6), (62) and (59) to obtain

$$W_0(\lambda,\mu)d\mu d\lambda = \sum_{k,l\geq 0}^{\infty} \frac{d\mu d\lambda}{\mu^{k+1}\lambda^{l+1}}\mathrm{res}_{\infty_+} x^k d\Omega_l(x) = \sum_{l\geq 0}\frac{d\lambda}{\lambda^{l+1}}d\tilde{\Omega}_l(\mu) \tag{86}$$

where $d\tilde{\Omega}_k = d\Omega_k - \frac{1}{2}dx^k$ is the meromorphic second-kind Abelian differential with the only singularity at ∞_-, where it behaves like

$$d\tilde{\Omega}_k(\mu) = -k\left(\mu^{k-1} + O(1)\right)d\mu, \text{ for } \mu \to \infty_-, \ k > 0 \tag{87}$$

and has vanishing A-periods $\oint_{A_i} d\tilde{\Omega}_k = 0$. Therefore, $W_0(\lambda,\mu)$ is holomorphic everywhere if both λ and μ correspond to the points on the same sheet, but it develops the second order pole at $\mu = \lambda$, where two points are located on different sheets. Indeed, taking into account the only non-holomorphic part gives

$$W_0(\lambda,\mu) \underset{\lambda\to\mu}{\sim} -\sum_k \frac{k\lambda^{k-1}}{\mu^{k+1}} = -\frac{1}{(\lambda-\mu)^2} \tag{88}$$

Besides, the evident normalizing condition, fixing the holomorphic part, immediately follows from (86),

$$\oint_{A_i} W_0(\lambda,\mu)d\mu = \oint_{A_i} W_0(\lambda,\mu)d\lambda = 0 \tag{89}$$

due to (72) and since $W_0(\lambda,\mu)$ is symmetric in λ and μ by definition.

Therefore, we finally come to the formula for the 2-point resolvent,

$$W_0(\lambda,\mu)d\lambda d\mu = \frac{\partial W_0(\lambda)}{\partial V(\mu)}d\lambda d\mu = -B(P,Q^*), \tag{90}$$

where we have introduced the $*$-involution between the two sheets of the hyperelliptic curve $\mathcal{C}$, so that Q^* denotes the image of Q under this involution. The only singularity of (90), for a fixed point P on a physical sheet, is at the point $Q \to P^*$ on the unphysical sheet with $\mu(Q) = \lambda(P^*) = \lambda(P)$, while on the other sheet it is cancelled under change of sign of $\tilde{y}$.

Therefore, in order to calculate the 2-point resolvent, one needs to write down the Bergmann bidifferential on the hyperelliptic curve manifestly. In principle, it has several different representations (one of the most hard for any further treatment is given by formula (5.20) in [10], borrowed from [46]). The simplest one can be obtained using formula (84). Indeed, a simple hyperelliptic representation of the Szegö kernel exists for the even non-singular half-integer characteristics. Such characteristics are in one-to-one correspondence with the partitions of the set of all the $2g+2$ ramification points into two equal subsets, $\{\mu_\alpha^+\}$ and $\{\mu_\alpha^-\}$, $\alpha = 1,\ldots,g+1$, $y_\pm(\lambda) \equiv \prod_{\alpha=1}^{g+1}(\lambda-\mu_\alpha^\pm)$, i.e. $y(\lambda) = y_+(\lambda)y_-(\lambda)$. Given these two sets, one can define $U_e(\lambda) = \frac{y_+(\lambda)}{y_-(\lambda)}$. In terms of these functions, the Szegö kernel is equal to [43, 47]

$$\Psi_e(\lambda,\mu) = \frac{U_e(\lambda)+U_e(\mu)}{2\sqrt{U_e(\lambda)U_e(\mu)}}\frac{\sqrt{d\lambda d\mu}}{\lambda-\mu} \tag{91}$$

Square of this expression, due to (84), leads to the manifest expression for the singular part of 2-point resolvent or the Bergmann kernel, i.e.

$$W_0(\lambda,\mu)d\lambda d\mu = \frac{y_+^2(\lambda)y_-^2(\mu) + y_+^2(\mu)y_-^2(\lambda) - 2y(\lambda)y(\mu)}{4y(\lambda)y(\mu)}\frac{d\lambda d\mu}{(\lambda-\mu)^2} + \text{holomorphic part} \tag{92}$$

where we choose the sign in front of $2y(\lambda)y(\mu)$ in the numerator so that the pole lies on the unphysical sheet. The holomorphic part is fixed now by the condition of zero A-periods.

However, in our further calculations we need another, completely different expression for the Bergmann kernel [34]. It can be most immediately obtained from the loop equation. The loop equation for the 2-point resolvent has the form (see, e.g., formula (I.3.40) in [20])

$$V'(\lambda)W_0(\lambda,\mu) - \hat{r}_V(\lambda)W_0(\mu) = 2W_0(\lambda)W_0(\lambda,\mu) + \frac{\partial}{\partial\mu}\frac{W_0(\lambda)-W_0(\mu)}{\lambda-\mu} \tag{93}$$

where the operator $\hat{r}_V(\lambda)$ is defined in (17), i.e. on a hyperelliptic curve $y^2 = R(x)$, e.g. (19), one gets

$$W_0(\lambda,\mu) = \frac{1}{y(\lambda)}\left[\frac{\partial}{\partial\mu}\frac{W_0(\lambda)-W_0(\mu)}{\lambda-\mu} + \hat{r}_V(\lambda)W_0(\mu)\right] = \tag{94}$$

$$= -\frac{1}{2(\lambda-\mu)^2} + \frac{y(\lambda)}{2y(\mu)}\left[\frac{1}{(\lambda-\mu)^2} - \frac{1}{2(\lambda-\mu)}\frac{\partial\log R(\lambda)}{\partial\lambda} - \frac{1}{4}\hat{r}_V(\mu)\log R(\lambda)\right]$$

One can check by straightforward calculation that this formula leads to a symmetric expression (see (III.2.6) in [20])

$$W_0(\lambda,\mu)d\lambda d\mu = \frac{V'(\mu)V'(\lambda) + \frac{1}{2}\left(\lambda Q(\mu) + \mu Q(\lambda)\right) + c - y(\mu)y(\lambda)}{2y(\lambda)y(\mu)}\frac{d\lambda d\mu}{(\lambda-\mu)^2} + \text{holomorphic part} \tag{95}$$

which is a particular case of (92), when parameterizing the hyperelliptic curve as

$$y^2(\lambda) = V'^2(\lambda) + \lambda Q(\lambda) + c \tag{96}$$

In the case of degenerate Riemann surface (21), one obtains (see [34]) similarly to (94)

$$W_0(\lambda,\mu)d\lambda d\mu = -\frac{d\lambda d\mu}{2(\lambda-\mu)^2} + \frac{\tilde{y}(\lambda)}{2\tilde{y}(\mu)}\left(\frac{1}{(\mu-\lambda)^2} + \frac{1}{2}\sum_{\alpha=1}^{2n}\left[\frac{1}{(\mu-\lambda)(\lambda-\mu_\alpha)} + \frac{\mathcal{L}_\alpha(\mu)}{\lambda-\mu_\alpha}\right]\right)d\lambda d\mu, \tag{97}$$

where $\mathcal{L}_\alpha(\mu) \equiv \sum_{l=0}^{n-2}\mathcal{L}_{\alpha,l}\mu^l$ are polynomials in μ. They can be unambiguously fixed by the requirements of absence of the first-order poles at $\lambda = \mu$ and zero A-periods.

2.3 Calculating $\mathcal{L}_\alpha(\mu)$

Although one has now the explicit formula for the 2-point resolvent (=Bergmann bidifferential) on the hyperelliptic surface in terms of branching points (97), it contains the polynomials $\mathcal{L}_\alpha(\mu)$ defined by the implicit requirements.

To find effective formulas for these polynomials, let us set $\lambda = \mu_\alpha$ in (97), and introduce the notation

$$\tilde{y}_\alpha(\lambda) \equiv \sqrt{\prod_{\beta\neq\alpha}(\lambda-\mu_\beta)}, \qquad \tilde{y}_\alpha \equiv \sqrt{\prod_{\beta\neq\alpha}(\mu_\alpha-\mu_\beta)} \tag{98}$$

We will also denote by square brackets the *fixed* argument of the Bergmann bi-differential (in some local coordinate), and consider $B(P,[\mu])$ as a 1-differential on $\mathcal{C}$. From (90), one has

$$-2B(P,[\mu_\alpha]) = \frac{1}{2}\frac{\mathcal{L}_\alpha(\lambda)}{\tilde{y}(\lambda)}\tilde{y}_\alpha^2 d\lambda + \frac{1}{2}\frac{\tilde{y}_\alpha^2}{(\lambda-\mu_\alpha)\tilde{y}(\lambda)}d\lambda \tag{99}$$

From (80), we have

$$\oint_{A_i}\frac{\mathcal{L}_\alpha(\mu)}{\tilde{y}(\mu)}d\mu = -\oint_{A_i}\frac{d\mu}{(\mu-\mu_\alpha)\tilde{y}(\mu)}. \tag{100}$$

and integrand in the l.h.s. is a linear combination of canonical holomorphic differentials $d\omega_i$ (51), so that

$$\mathcal{L}_\alpha(\mu) = -\sum_{i=1}^{n-1}H_i(\mu)\cdot\oint_{A_i}\frac{d\lambda}{(\lambda-\mu_\alpha)\tilde{y}(\lambda)}, \tag{101}$$

and, in particular,

$$\mathcal{L}_\alpha(\mu_\alpha) = -\sum_{i=1}^{n-1}H_i(\mu_\alpha)\cdot\oint_{A_i}\frac{d\lambda}{(\lambda-\mu_\alpha)\tilde{y}(\lambda)} \tag{102}$$

Another equivalent representation we will need in sect. 4, is

$$\sum_{l=0}^{n-2} \mathcal{L}_{\alpha,l}\mu_\alpha^l = -\sum_{j=1}^{n-1} \oint_{A_j} \frac{H_j(\lambda)}{(\lambda-\mu_\alpha)\tilde{y}(\lambda)} d\lambda, \quad \alpha = 1,\ldots,2n. \tag{103}$$

Indeed, let us introduce the quantities

$$\sigma_{j,i} \equiv \oint_{A_j} \frac{\lambda^{i-1}}{\tilde{y}(\lambda)} d\lambda, \quad i,j = 1,\ldots,n-1 \tag{104}$$

Then, for the canonical polynomials $H_k(\lambda) \equiv \sum_{l=1}^{n-1} H_{l,k}\lambda^{l-1}$, $k = 1,\ldots,n-1$, related to the canonically normalized differentials (51), i.e. $\oint_{A_j} \frac{H_k(\lambda)}{\tilde{y}(\lambda)} d\lambda = \delta_{k,j}$, one obviously has

$$\sum_{l=1}^{n-1} \sigma_{j,l} H_{l,k} = \delta_{j,k} \quad \text{for} \quad j,k = 1,\ldots,n-1. \tag{105}$$

Therefore, for all $k > 0$ such that $j - 1 - k \geq 0$,

$$\sum_{i=1}^{n-1} H_{j,i} \cdot \oint_{A_i} \frac{\lambda^{j-k-1}}{\tilde{y}(\lambda)} d\lambda = 0 \tag{106}$$

Then,

$$\begin{aligned}
&\sum_{i=1}^{n-1} \oint_{A_i} \frac{H_i(\lambda) - H_i(\mu_\alpha)}{(\lambda-\mu_\alpha)\tilde{y}(\lambda)} d\lambda = \\
&= \sum_{i=1}^{n-1} \oint_{A_i} \frac{\sum_{j=2}^{n-1} H_{j,i} \sum_{k=1}^{j-1} \lambda^{j-1-k}\mu_\alpha^{k-1}}{\tilde{y}(\lambda)} d\lambda = 0.
\end{aligned}$$

and, because of (102), we finally arrive at (103). The above formulas mean that for the Bergmann kernel on hyperelliptic curve $y^2 = R(x)$ one can write

$$\begin{aligned}
&W_0(\lambda,\mu)d\lambda d\mu = \\
&-\frac{d\lambda d\mu}{2(\lambda-\mu)^2} + \frac{y(\lambda)}{2y(\mu)}\left(\frac{1}{(\mu-\lambda)^2} + \frac{1}{2}\sum_{\alpha=1}^{2n}\left[\frac{1}{(\mu-\lambda)(\lambda-\mu_\alpha)} - \sum_{i=1}^{n-1} H_i(\mu)\oint_{A_i}\frac{dx}{(x-\mu_\alpha)^2 y(x)}\right]\right) d\lambda d\mu = \\
&= -\frac{d\lambda d\mu}{2(\lambda-\mu)^2}\left(1 - \frac{R(\lambda)}{y(\lambda)y(\mu)}\right) - \frac{d\lambda d\mu}{2(\lambda-\mu)}\frac{R'(\lambda)}{y(\lambda)y(\mu)} - \frac{1}{2}y(\lambda)d\lambda\sum_{\alpha=1}^{2n}\sum_{i=1}^{n-1} d\omega_i(\mu)\oint_{A_i}\frac{dx}{(x-\mu_\alpha)^2 y(x)}
\end{aligned} \tag{107}$$

A particular case of this formula at $\lambda = 0$ was used in [48] for solving the quasiclassical Bethe anzatz equations in the context of AdS/CFT correspondence. The last term in the r.h.s. of (107) is explicit form for the action of the operator (17) in the case of smooth Riemann surface, restricted by the vanishing period's constraints.

2.4 Mixed second derivatives

Another set of relations follows from the general properties of the Bergmann kernel, and can be also derived directly from the formulas of sect. 1. To this end, we apply the mixed derivatives $\partial/\partial V(\mu)$ and $\partial/\partial S_i$ to the planar limit of the free energy $\mathcal{F}_0$. On one hand, $\partial\mathcal{F}_0/\partial S_i = \oint_{B_i} dS$ and using that $dS(\lambda) = y(\lambda)d\lambda = (V'(\lambda) - 2W_0(\lambda))d\lambda$, $\frac{\partial V'(\lambda)}{\partial V(\mu)} = -\frac{1}{(\lambda-\mu)^2}$ and formula (90), one obtains that

$$\begin{aligned}
\oint_{B_i} \frac{\partial(dS(\lambda))}{\partial V(\mu)} &= \oint_{B_i}\left(2B(P,[\mu]) - \frac{1}{(\lambda-\mu)^2}dp\right) = \\
&= \oint_{B_i} 2B(P,[\mu]).
\end{aligned}$$

On the other hand, acting by derivatives in the opposite order, one first obtains $\partial\mathcal{F}_0/\partial V(\mu) = W_0(\mu) = V'(\mu) - y(\mu)$ and then $\partial(V'(\mu) - y(\mu))/\partial S_i = 2d\omega_i([\mu])$, or, in the coordinate-free notation, one of the Fay identities [43]:

$$\frac{1}{2\pi i}\oint_{B_i} B(P,Q) = d\omega_i(Q) \tag{108}$$

This means that

$$\frac{\partial dS(\mu)}{\partial S_i} = \left[\oint_{B_i} \frac{\partial dS(\lambda)}{\partial V(\mu)} d\lambda\right] d\mu = \oint_{B_i} \frac{\partial dS(\mu)}{\partial V(\lambda)} d\lambda \tag{109}$$

where the *both* integrals are taken over the variable λ. Now, as $dS(\mu) = y(\mu)d\lambda$ is the generating function for the variables $\xi_a \equiv \{\mathcal{M}, \{\lambda_i\}, \{\mu_\alpha\}, M_\alpha^{(i)}\}$, given by (21), (23), and (25) ($M_\alpha^{(i)}$ are just ith order derivatives of $M_{m-n}(\mu)$ at $\mu = \mu_\alpha$) giving their dependence on S_i and t_k, one concludes that similar relation for the first derivatives holds for each of these variables. Indeed, multiplying (109) by $\frac{1}{y(\mu)}$ one then can bring μ successively to μ_α's, λ_i's and ∞ to pick up pole terms with different ξ_a and prove that

$$\frac{\partial \xi_a}{\partial S_i} = \oint_{B_i} \frac{\partial \xi_a}{\partial V(\lambda)} d\lambda \tag{110}$$

As a consequence, any function G of ξ_a would naively satisfy the same relation. However, there is a subtle point here: in formula (108) one could integrate the both sides over an A_j-cycle to obtain at the r.h.s. δ_{ij}, while the A-periods of the Bergmann kernel are zero. It means that one should carefully permute the integrations, since A_i- and B_i-cycles intersect. Therefore, one should carefully take into account the contributions of the intersection points (A_i- and B_j-cycles for different i and j *do not* intersect and integrations can be exchanged, which perfectly match the Kronecker symbol obtained) which does not vanish due to the double pole of the Bergmann kernel.

In particular, we formally have

$$\frac{\partial M(\lambda)}{\partial S_i} = \oint_{B_i} d\mu \left(\frac{\partial M(\lambda)}{\partial V(\mu)} + \frac{\partial}{\partial\mu}\frac{1}{(\lambda-\mu)\tilde{y}(\mu)}\right), \tag{111}$$

where the second term in the brackets vanishes unless we have an outer integration over the cycle A_i w.r.t. the variable λ in the both sides.

Similarly, one should take care when omitting the second term in the second equality in (108). Indeed, suppose we consider S_i as a function of ξ_a. Then,

$$\frac{\partial S_i(\xi_a)}{\partial S_j} \neq \oint_{B_i} \frac{\partial S_i}{\partial V(\lambda)} \tag{112}$$

Here again one should note that S_i is the integral of dS over the A_i cycle and take care when exchanging integrations. While doing this, the contributions from the omitted term in (108) are to be taken into account. Note, however, that one can *never* express S_i as a function of a *finite* number of "local" variables ξ_a.

Therefore, if one considers the functions G that depends on only finite number of "local" variables ξ_a (the branching points, $M_\alpha^{(i)}$—the moments of the model, and, possibly, zeros of the polynomial $M(x)$), the relation

$$\frac{\partial G}{\partial S_i} = \oint_{B_i} \frac{\partial G}{\partial V(\lambda)} d\lambda \tag{113}$$

holds, while, for A-cycle integrals over the Riemann surface (note that the B-cycle integrals do not meet such a problem), one should add more terms in the r.h.s. in order to take into account the second, pole term in (108), (111).

Relation (113) is therefore valid in *all* orders of $1/N$-expansion for the 1MM free energy because any higher-genus contribution is a function only of μ_α and of a *finite* number of higher moments $M_\alpha^{(i)}$.

Equation (97) implies, for $\lambda \sim \mu_\alpha \sim \mu$, in local coordinates $d\lambda/\sqrt{\lambda-\mu_\alpha} = 2d\left(\sqrt{\lambda-\mu_\alpha}\right)$ and $d\mu/\sqrt{\mu-\mu_\alpha} = 2d\left(\sqrt{\mu-\mu_\alpha}\right)$ the relation

$$\mathcal{L}_\alpha(\mu_\alpha) = B([\mu_\alpha],[\mu_\alpha])|_{nonsing.} = S_B(\mu_\alpha). \tag{114}$$

Applying now (108) and (103), one immediately comes to

$$\mathcal{L}_\alpha(\mu_\alpha) = S_B(\mu_\alpha) = \sum_{i=1}^{n-1} \oint_{A_i}\oint_{B_i} \frac{B(P,Q)}{\lambda-\mu_\alpha} \tag{115}$$

in the local coordinates associated with the hyperelliptic Riemann surface (21).

3 WDVV equations

The general form of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [24, 25] is the systems of algebraic equations [26]

$$\mathcal{F}_I\mathcal{F}_J^{-1}\mathcal{F}_K = \mathcal{F}_K\mathcal{F}_J^{-1}\mathcal{F}_I, \quad \forall\ I,J,K \tag{116}$$

on the third derivatives

$$\|\mathcal{F}_I\|_{JK} = \frac{\partial^3\mathcal{F}}{\partial t_I\,\partial t_J\,\partial t_K} \equiv \mathcal{F}_{IJK} \tag{117}$$

of some function $\mathcal{F}(\{t_I\})$. These equations often admit an interpretation as associativity relations in some algebra (of polynomials, differentials etc.) and are relevant for describing topological theories.

As WDVV systems are often closely related to Whitham systems, a natural question is whether the corresponding 1MM free-energy function satisfy the WDVV equations? This was proved in [31], where it was shown that the multicut solution the 1MM satisfies the WDVV equations as a function of *canonical* variables identified with the periods and residues of the generating meromorphic one-form dS [9]. The method to prove it consists of two steps. The first, most difficult, step is to find the residue formula for the third derivatives (117) of the 1MM free energy. Then, using an associativity, one immediately proves that the free energy of multi-support solution satisfies the WDVV equations if the number of independent variables is *equal* to the number of branching (critical) points in the residue formula. We show here that the statement holds in the case of arbitrary potentials for a fixed-genus reduced Riemann surface.

In sect. 3.1, we derive the residue formula for the third derivatives of the quasiclassical tau-function for the variables (the generalized times) t_I associated with both the periods S_i and residues t_k of the generating differential dS. In sect. 3.2, we prove that the free energy of the multi-interval-support solution $\mathcal{F}_0(t_I)$ solves WDVV equations (116) as a function of the subset $\{t_\alpha\} \subseteq \{t_I\}$; the total number of t_α must be fixed to be equal to the number of branching points in the residue formula for the third derivatives (117) in order to make the set of the WDVV equations nontrivial.

3.1 Residue formula

Because all the quantities $d\Omega_I$ (78) depend *entirely* on the reduced hyperelliptic Riemann surface (21), their derivatives w.r.t. *any parameter* must be expressed through the derivatives w.r.t. the positions of the branching points μ_α. So, calculating derivatives w.r.t. μ_α is the basic ingredient. Note that although the differentials $d\Omega_I$ are regular at the points μ_α in the local coordinate $d\tilde{y} \sim d\lambda/\sqrt{\lambda-\mu_\alpha}$, the derivatives $\partial d\Omega_I/\partial\mu_\alpha$ obviously develop singularities at $\lambda=\mu_\alpha$, and we must bypass these singularities when choosing the integration contour as in Fig. 2.

Let us now derive the formulas for the third derivatives $\partial^3\mathcal{F}_0/(\partial t_I\partial t_J\partial\mu_\alpha) \equiv \mathcal{F}_{IJ\alpha}$, following [9, 31]. Consider, first, the case where the "times" t_I and t_J are S_i and S_j, and $\frac{\partial^2\mathcal{F}_0}{\partial t_I\partial t_J} \equiv \mathcal{F}_{IJ} = T_{ij}$. We note that the derivatives of the elements of period matrix can be expressed through the integral over the

"boundary" $\partial\Sigma_g$ of the cut Riemann surface Σ_g (see Fig. 2), Indeed, because of the normalization condition $\oint_{A_l} d\omega_j = \delta_{ij}$, we have $\oint_{A_l} \partial_\alpha d\omega_j = 0$, so that

$$\begin{aligned}
\frac{\partial T_{ij}}{\partial \mu_\alpha} &= \oint_{B_j} \partial_\alpha d\omega_i = \sum_{l=1}^{g} \left(\oint_{A_l} d\omega_j \oint_{B_l} \partial_\alpha d\omega_i - \oint_{B_l} d\omega_j \oint_{A_l} \partial_\alpha d\omega_i \right) = \\
&= \sum_{l=1}^{g} \left(\int_{B_l} \omega_j^+ \partial_\alpha d\omega_i - \int_{B_l} \omega_j^- \partial_\alpha d\omega_i \right) - \\
&\qquad - \sum_{l=1}^{g} \left(\int_{A_l} \omega_j^+ \partial_\alpha d\omega_i - \int_{A_l} \omega_j^- \partial_\alpha d\omega_i \right) = \\
&= \oint_{\partial\Sigma_g} \omega_j \partial_\alpha d\omega_i
\end{aligned} \tag{118}$$

where $\omega_j = \int d\omega_j$ are Abelian integrals and we let $\omega_j^\pm$ denote their values on two copies of cycles on the cut Riemann surface in Fig. 2.

We must now choose the cycles A_l and B_l bypassing all possible singularities of the integrand (in this case, the ramification points μ_β). Expression (118) can be then evaluated through the residue formula

$$\partial_\alpha T_{ij} = - \int_{\partial\Sigma_g} \omega_j \partial_\alpha d\omega_i = \sum \mathrm{res}_{d\lambda=0} \left(\partial_\alpha \omega_j d\omega_i \right). \tag{119}$$

The proof of this formula for generic variation of moduli can be found in [31]. Here we will adjust it for the class of variations in terms of the branch points $\{\mu_\alpha\}$, i.e. to the class of hyperelliptic Riemann surfaces.

Before evaluating this sum of residues, let us consider the case of meromorphic differentials. Then, using formulas (62) and (77), one obtains

$$\frac{\partial^2 \mathcal{F}_0}{\partial t_k \partial t_l} = \frac{1}{2} \oint_{C_L} \left((\Omega_k)_{+,0} d\Omega_l \right), \quad k, l \geq 0, \tag{120}$$

where $(\Omega_k)_{+,0}$ is the singular part of Ω_k at infinity, i.e., it is λ^k for $k > 0$ and the logarithmic function for $k = 0$. Because $\partial(\Omega_k)_{+,0}/\partial\mu_\alpha = 0$ for $k \geq 0$, we have $\partial d\Omega_k/\partial\mu_\alpha = \partial (d\Omega_k)_-/\partial\mu_\alpha$, where $(d\Omega_k)_-$ is the holomorphic part of $d\Omega_k$ at infinity (and the expression $(\Omega_k)_-$ is therefore meromorphic for all $k \geq 0$). We then have

$$\begin{aligned}
\frac{\partial}{\partial \mu_\alpha} \frac{1}{2} \oint_{C_L} \left((\Omega_k)_{+,0} d\Omega_l \right) &= \frac{1}{2} \oint_{C_L} \left((\Omega_k)_{+,0} \partial_\alpha (d\Omega_l)_- \right) = \\
= -\frac{1}{2} \oint_{C_L} \left(d\Omega_k \partial_\alpha (\Omega_l) \right) &= -\mathrm{res}_\infty \left(d\Omega_k \partial_\alpha \Omega_l \right).
\end{aligned} \tag{121}$$

The last expression can be rewritten as

$$\begin{aligned}
-\mathrm{res}_\infty \left(d\Omega_k \frac{\partial \Omega_l}{\partial \mu_\alpha} \right) &= \oint_{\partial\Sigma_g} \left(d\Omega_k \frac{\partial \Omega_l}{\partial \mu_\alpha} \right) + \sum_{\beta=1}^{2n} \mathrm{res}_{\mu_\beta} \left(d\Omega_k \partial_\alpha \Omega_l \right) = \\
&= \sum_{\beta=1}^{2n} \mathrm{res}_{\mu_\beta} \left(d\Omega_k \partial_\alpha \Omega_l \right)
\end{aligned} \tag{122}$$

because $\oint_{\partial\Sigma_g} \left(d\Omega_k \partial_\alpha \Omega_l \right) = 0$ following the same arguments as in formula (118) and due to normalization conditions (72).

Further computations for both holomorphic and meromorphic differentials coincide as we need only their local behavior at the vicinity of a point $\lambda = \mu_\beta$. Using explicit expression (78), we have

$$\partial_\alpha \Omega_J = \frac{H_J(\mu_\alpha)}{\prod_{\gamma\neq\alpha} \sqrt{\mu_\alpha - \mu_\gamma}} (\lambda - \mu_\alpha)^{-1/2} + O(\sqrt{\lambda - \mu_\alpha}), \tag{123}$$

for $\beta = \alpha$, and $\partial_\alpha \Omega_J \sim \sqrt{\lambda - \mu_\beta}$ otherwise. Together with (78), this means that the only point to evaluate the residue is $\lambda = \mu_\alpha$ at which we have (cf. [43])

$$\mathcal{F}_{0,IJ\alpha} = \mathrm{res}_{\mu_\alpha}(d\Omega_I \partial_\alpha \Omega_J) = \frac{H_I(\mu_\alpha)H_J(\mu_\alpha)}{\prod_{\beta\neq\alpha}(\mu_\alpha - \mu_\beta)}. \tag{124}$$

Completing the calculation of the third derivative needs just inverting the dependence on the ramification points therefore finding $\partial\mu_\alpha/\partial t_K$. Differentiating expressions (19), (21) w.r.t. t_K for computing (59) we obtain

$$\begin{aligned}\frac{\partial dS}{\partial t_K} &= \frac{H_K(\lambda)d\lambda}{\tilde{y}(\lambda)} = \\ &= \frac{1}{2}M_{m-n}(\lambda)\sum_{\alpha=1}^{2n}\frac{\tilde{y}(\lambda)}{(\lambda-\mu_\alpha)}\frac{\partial\mu_\alpha}{\partial t_K}d\lambda + \frac{\partial M_{m-n}(\lambda)}{\partial t_K}\tilde{y}(\lambda)d\lambda.\end{aligned} \tag{125}$$

The derivative of the polynomial $M_{m-n}(\lambda)$ is obviously polynomial and regular at $\lambda = \mu_\alpha$. Multiplying (125) by $\sqrt{\lambda-\mu_\alpha}$ and setting $\lambda = \mu_\alpha$, we immediately obtain

$$\frac{\partial\mu_\alpha}{\partial t_K} = \frac{H_K(\mu_\alpha)}{M_{m-n}(\mu_\alpha)\prod_{\beta\neq\alpha}(\mu_\alpha-\mu_\beta)}. \tag{126}$$

Combining this with (124), we come to the desired residue formula for the third derivative w.r.t. the canonical variables t_I:

$$\frac{\partial^3\mathcal{F}_0}{\partial t_I\partial t_J\partial t_K} = \sum_{\alpha=1}^{2n}\frac{H_I(\mu_\alpha)H_J(\mu_\alpha)H_K(\mu_\alpha)}{M_{m-n}(\mu_\alpha)\prod_{\beta\neq\alpha}(\mu_\alpha-\mu_\beta)^2} = \sum_\alpha \mathrm{res}_{\mu_\alpha}\frac{d\Omega_I d\Omega_J d\Omega_K}{d\lambda dy} = \mathrm{res}_{d\lambda=0}\frac{d\Omega_I d\Omega_J d\Omega_K}{d\lambda dy} \tag{127}$$

3.2 Proof of WDVV equations

Given residue formula (127), the proof of WDVV equations (116) can be done, following [26]-[29], by checking associativity of the algebra of differentials $d\Omega_I$ with multiplication modulo $\tilde{y}(\lambda)d\lambda$. This algebra is reduced to the algebra of polynomials $H_I(\lambda)$ with multiplication modulo $\tilde{y}^2(\lambda)$ which is correctly defined and associative. The basis of the algebra of $H_I(\lambda)$ obviously has dimension $2n$ and is given, e.g., by monomials of the corresponding degrees $0, 1, \ldots, 2n-2, 2n-1$. The study of such algebras can be performed even for non-hyperelliptic curves, the details can be found in [28, 49].

Another proof is even more simple and reduces to solving the system of linear equations [50, 30]. To this end, we first define

$$\phi_I^\alpha \equiv \frac{H_I(\mu_\alpha)}{M_{m-n}^{1/3}(\mu_\alpha)\prod_{\beta\neq\alpha}(\mu_\alpha-\mu_\beta)^{2/3}} \tag{128}$$

so that (127) can be rewritten as

$$\mathcal{F}_{0,IJK} = \sum_\alpha \phi_I^\alpha\phi_J^\alpha\phi_K^\alpha \tag{129}$$

Now let us fix some index Y and consider the following multiplication

$$\phi_I^\alpha\phi_J^\alpha = \sum_K C_{IJ}^{(Y)K}\phi_K^\alpha\phi_Y^\alpha, \quad \forall\ \alpha \tag{130}$$

the structure constants $C_{IJ}^{(Y)K}$ being independent of α. One can equally look at this as at a system of *linear equations* for C_{IJ}^K at fixed values of I and J. If this system has a solution, (130) gives rise to an associative ring, with the structure constants C_{IJ}^K satisfying (associativity condition)

$$\left(C_I^{(Y)}\right)_L^K\left(C_J^{(Y)}\right)_M^L = \left(C_J^{(Y)}\right)_L^K\left(C_I^{(Y)}\right)_M^L, \quad (C_I^{(Y)})_J^K \equiv C_{IJ}^{(Y)K} \tag{131}$$

Now, the solution to (130) is

$$C_{IJ}^{(Y)K} = \sum_{\alpha} \phi_I^\alpha \phi_J^\alpha \left(\Phi^\alpha_{(Y)K} \right) \tag{132}$$

where Φ_K^α is the matrix[18] inverse to $\phi_K^\alpha \phi_Y^\alpha$. This solution exists if the number of vectors (variables K) is *greater or equal* the number $2n$ of the branching points μ_α.

The other important condition is the *invertibility* of the matrix $\phi_K^\alpha \phi_Y^\alpha$, or the matrix ϕ_K^α (we suppose $\phi_Y^\alpha \neq 0$) which ensures the nontriviality of the WDVV relations. For this, we must require the number of vectors ϕ_I to be *less or equal* the number $2n$ of their components. We therefore obtain the following two conditions [30]:

- the "matching" condition
$$\#(I) = \#(\alpha); \tag{133}$$
  and
- the nondegeneracy of the matrix ϕ_I^α (see the proof in sect. 4.3):
$$\det_{I\alpha} \|\phi_I^\alpha\| \neq 0 \tag{134}$$

Now, using (130), one rewrites (129)

$$\mathcal{F}_{0,IJK} = \sum_{\alpha,L} C_{IJ}^{(Y)L} \phi_K^\alpha \phi_L^\alpha \phi_Y^\alpha = \sum_L C_{IJ}^{(Y)L} \mathcal{F}_{0,KLY} \tag{135}$$

coming to the matrix formula that express the structure constants through the third derivatives of the planar limit free energy

$$C_I^{(Y)} = \mathcal{F}_I \mathcal{F}_Y^{-1} \tag{136}$$

where we denoted $\mathcal{F}_I$ the matrix with element JK equal to $\mathcal{F}_{0,IJK}$. Now substituting (136) into (131), one immediately arrives at (116).

Thus, we established that conditions (116) require the number of varying parameters $\{t_I\}$ to satisfy matching condition (133). We let $\{t_\alpha\}$ denote these "primary" variables. It is convenient to set classical "primary" variables w.r.t. which WDVV equations (116) hold true, to be the parameters S_i, $i = 1, \ldots, n-1 \equiv g$, t_0, and t_k with $k = 1, \ldots, n$ keeping all other times frozen.

Below, in sect. 4.3, we interpret the answer in genus one in terms of the determinant relation for the third derivatives $\mathcal{F}_{IJK}$.

4 Higher genus contributions

The solution $W_1(\lambda)$ to the loop equations in the multicut case was first found by Akemann [34].[19] He also managed to integrate them in to obtain the free energy $\mathcal{F}_1$ in the two-cut case. The genus-one partition function in the generic multi-cut case was proposed in [51, 52], where it was observed that the Akemann formula coincides with the correlator of twist fields, computed by Al.Zamolodchikov [53]. This produces cuts on complex plane and gives rise to a hyperelliptic Riemann surface, following the ideology of [45], some corrections to this construction are due to the star operators, introduced in [54]. In this section, we present (see also [55]) the derivation of the genus-one correction based on solving the loop equation, and generalizing Akemann's result for the partition function to arbitrary number of cuts.

[18] We consider it as a matrix of indices K and α, while the reference index Y is implied as a silent parameter of the whole consideration.

[19] The universal critical behavior of the corresponding correlation functions was discussed in [5].

4.1 The iterative procedure

Iterative solving the loop equations. Thus, now we are going to determine higher genus contributions. We do this iteratively by inverting the genus expanded loop equation (11). Our strategy will be to construct an integral operator $\widehat{d\mathcal{G}}$ inverse to the integral operator $\widehat{K} - 2W_0(\lambda)$.

Acting with this operator onto the both sides of the loop equation eq.(11), one recursively produces $W_h(\lambda)$ for all genera like all the multi-point resolvents of the same genus can be obtained from $W_h(\lambda)$ merely applying the loop insertion operator $\frac{\partial}{\partial V(\lambda)}$.

However, there is a subtlety: the operator $\widehat{K}-2W_0(\lambda)$ has zero modes and is not invertible. Therefore, solution to the loop equation is determined up to an arbitrary combination of these zero modes. Hence, the kernel of the operator $\widehat{K} - 2W_0(\lambda)$ is spanned exactly by holomorphic one-differentials on the Riemann surface (21).

In order to fix this freedom, we assume $W_h(\lambda)$ is expressed exclusively in terms of derivatives $\frac{\partial \mu_\alpha}{\partial V(\lambda)}$ and $\frac{\partial M_\alpha^{(k)}}{\partial V(\lambda)}$, which, as we show in the next paragraph, fixes a solution to the loop equation. It is a natural extension of the normalizing property (60) to higher genera and can be ultimately written in the form

$$\oint_{A_i} \frac{\partial \mathcal{F}_h}{\partial V(\lambda)} d\lambda \equiv \oint_{A_i} W_h(\lambda) d\lambda = 0 \quad \forall i \text{ and for } h \geq 1. \tag{137}$$

Now we claim that the integral operator

$$\widehat{d\mathcal{G}}\,(f)\,(\lambda) \equiv \oint_{\mathcal{C}_{\mu_\alpha}} \frac{d\mu}{2\pi i} \frac{d\mathcal{G}(\lambda,\mu)}{d\lambda} \frac{1}{y(\mu)} \cdot f(\mu) \tag{138}$$

is an inverse for the operator $\widehat{K} - 2W_0(\lambda)$ in the space of rational functions $f(\mu)$ with poles at the points μ_α only.

Here the one-differential $d\mathcal{G}(\lambda,\mu)$ w.r.t. the first argument λ [20] is the primitive of the Bergmann kernel $B(\lambda,\mu)$ w.r.t. the argument μ. Obviously, it is a single-valued differential of λ with zero A-periods on the reduced Riemann surface and is multiple-valued function of μ, which undergoes jumps equal to $d\omega_i(\lambda)$ when the variable μ passes through the cycle B_i (cf. with (107)):

$$d\mathcal{G}(\lambda,\mu) = \frac{\tilde{y}(\mu) d\lambda}{(\lambda-\mu)\tilde{y}(\lambda)} - \sum_{i=1}^{n-1} \frac{H_i(\lambda) d\lambda}{\tilde{y}(\lambda)} \oint_{A_i} d\xi \frac{\tilde{y}(\mu)}{(\xi-\mu)\tilde{y}(\xi)}. \tag{139}$$

Contours of integration over the cycles A_i must lie outside the contour of integration $\mathcal{C}_{\mu_\alpha}$ encircling the branching point μ_α in (150).

Moreover, this operator obeys the property

$$\oint_{A_i} \widehat{d\mathcal{G}}(f)(\lambda) d\lambda \equiv 0 \tag{140}$$

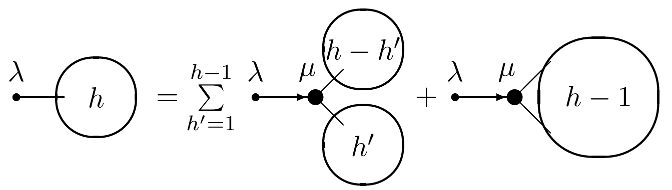
and, therefore, respects condition (137). Therefore, one has to solve the loop equations inverting $\widehat{K} - 2W_0(\lambda)$ exactly with $\widehat{d\mathcal{G}}$.

Then, the calculation immediately validates the diagrammatic technique [32] for evaluating multipoint resolvents in 1MM. Indeed, representing $d\mathcal{G}(\lambda,\mu)$ (for $\lambda > \mu$) as the arrowed propagator, the three-point vertex as dot in which we assume the integration over μ: $\bullet \equiv \oint \frac{d\mu}{2\pi i} \frac{1}{y(\mu)}$, we can graphically write solution to (11) since

$$W_h(\lambda) = \widehat{d\mathcal{G}} \left[\sum_{h'=1}^{h-1} W_{h'}(\cdot) W_{h-h'}(\cdot) + W_{h-1}(\cdot,\cdot) \right] (\lambda). \tag{141}$$

[20] It is the function $dS(\lambda,\mu)$ in the notation of [32].

Then, representing multiresolvent $W_{h'}(\lambda_1, \ldots, \lambda_k)$ as the block with k external legs and with the index h', one obtains



(142)

which is just the basis relation for the diagrammatic representation (141). Here, by convention, all integration contours for the variables μ lie inside each other in the order, established by arrowed propagators $d\mathcal{G}(\mu_i, \mu_j)$. The other, nonarrowed propagators are $W(\lambda, \mu) \equiv \frac{\partial}{\partial \mu} d\mathcal{G}(\lambda, \mu)$. All A-cycle contours of integration in $d\mathcal{G}(\lambda, \mu)$ are outside the contours of internal integrations over μ_i-variables.

Choosing a specific basis. In order to prove the claims of the previous paragraph, first of all, we change variables from coupling constants to special moment functions which allows one to apply higher genus machinery nonperturbatively in coupling constants t_j. This machinery turns out to involve only on a finite number of the moments $M_\alpha^{(k)}$ (24). (Recall that (23)-(24) implies $M_\alpha^{(1)} = M(\mu_\alpha)$.)

Now let us fix the generic analytic structures of the 1- and 2-point resolvents. First of all, note that $W_0(\lambda, \mu)$ in (94) is invariant w.r.t. the involution $y \to -y$ that permutes physical and unphysical sheets. Moreover, $W_0(\lambda, \lambda)$ is a fractional rational function of λ with poles at the points μ_α only. Further, look at the loop equation, (11) and rewrite it as

$$y(\lambda) W_h(\lambda) = \left[V'(\lambda) W_h(\lambda)\right]_+ + \sum_{h'=1}^{h-1} W_{h'}(\lambda) W_{h-h'}(\lambda) + \frac{\partial}{\partial V(\lambda)} W_{h-1}(\lambda), \tag{143}$$

It follows from this formula that the 1-point resolvent $W_1(\lambda)$ is also fractional rational function of λ with poles at the points μ_α only divided by $y(\lambda)$, i.e. it changes sign under permuting the physical and unphysical sheets. This procedure can be iterated with the loop equation written in the form [20]

$$V'(\lambda) W_h(\lambda, \lambda_1, ..., \lambda_n) = \hat{r}_V(\lambda) W_h(\lambda_1, ..., \lambda_n) + \sum_{h'=0}^{h} \sum_{n_1+n_2=n-1} W_{h'}(\lambda, \lambda_1, ..., \lambda_{n_1}) W_{h-h'}(\lambda, \lambda_1, ..., \lambda_{n_2}) +$$

$$+ \sum_i \frac{\partial}{\partial \lambda_i} \frac{W_h(\lambda, \lambda_1, ..., \check{\lambda}_i, ..., \lambda_n) - W_h(\lambda_1, ..., \lambda_n)}{\lambda - \lambda_i} + \frac{\partial}{\partial V(\lambda)} W_{h-1}(\lambda, \lambda_1, ..., \lambda_n) \tag{144}$$

In particular, using the known analytic structure of $W_0(\lambda, \mu)$ one easily checks that the 3-point resolvent $W_0(\lambda, \mu, \nu)$ is odd w.r.t. to the involution $y \to -y$ and then, with the knowledge of structure of $W_1(\lambda)$ and $W_0(\lambda, \mu)$, one can use the loop equation (144) to prove that $W_1(\lambda, \mu)$ is even w.r.t. the involution etc. The final result is that all $(2n)$-point resolvents W_h are even, while all $(2n+1)$-point resolvents W_h but $W_0(\lambda)$ are odd w.r.t. the involution. This means that all $W_h(\lambda)$ but $W_0(\lambda)$ are fractional rational functions of λ_i with poles at the points μ_α only divided by $y(\lambda)$. Moreover, the r.h.s. of eq. (11) is similarly a fractional rational function of λ having poles at μ_α only, i.e. one should naturally choose a specific basis $\chi_\alpha^{(k)}(\lambda)$ defined by the property that, for the integral operator in eq.(11),

$$(\widehat{K} - 2W_0(\lambda))\chi_\alpha^{(k)}(\lambda) = \frac{1}{(\lambda - \mu_\alpha)^k},$$
$$k = 1, 2, \ldots, \quad \alpha = 1, \ldots, 2n\,. \tag{145}$$

Then, $W_h(\lambda)$ must have the structure

$$W_h(\lambda) = \sum_{k=1}^{3h-1} \sum_{\alpha=1}^{2n} A_{\alpha,h}^{(k)} \chi_\alpha^{(k)}(\lambda), \quad h \geq 1, \tag{146}$$

where $A^{(k)}_{\alpha,h}$ are certain functions of μ_β and the moments $M^{(k)}_\beta$. As the order of the highest singularity term $1/((\lambda-\mu_\alpha)^{3h-1}\tilde{y}(\lambda))$ in $W_h(\lambda)$ is insensitive to a multi-cut structure[21], $W_h(\lambda)$ will depend on at most $2n(3h-2)$ moments, just like the one-cut solution case [35].

One could define a set of basis functions $\chi^{(k)}_\alpha(\lambda)$ recurrently, as in [35], [34], however, here we present another technique inspired by [32]. We first calculate the quantities $\frac{\partial\mu_\alpha}{\partial V(\lambda)}$ and $\frac{\partial M^{(k)}_\alpha}{\partial V(\lambda)}$.

Using the identity $\frac{\partial}{\partial V(\lambda)}V'(\mu) = -\frac{1}{(\lambda-\mu)^2}$ and representation (24), one easily obtains

$$\begin{aligned}
\frac{\partial M^{(k)}_\alpha}{\partial V(\lambda)} &= (k+1/2)\left(M^{(k+1)}_\alpha\frac{\partial\mu_\alpha}{\partial V(\lambda)} - \frac{1}{(\lambda-\mu_\alpha)^{k+1}\tilde{y}(\lambda)}\right)\\
&\quad + \frac{1}{2}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{2n}\sum_{l=1}^{k}\frac{1}{(\mu_\beta-\mu_\alpha)^{k-l+1}}\Big(\frac{1}{(\lambda-\mu_\alpha)^l\tilde{y}(\lambda)} - M^{(l)}_\alpha\frac{\partial\mu_\beta}{\partial V(\lambda)}\Big)\\
&\quad + \frac{1}{2}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{2n}\frac{1}{(\mu_\beta-\mu_\alpha)^k}\Big(M^{(1)}_\beta\frac{\partial\mu_\beta}{\partial V(\lambda)} - \frac{1}{(\lambda-\mu_\beta)\tilde{y}(\lambda)}\Big)\\
&\quad \alpha = 1,\ldots,2n\ ,\ \ k=1,2,\ldots .
\end{aligned}\tag{147}$$

Note that the general structure of this formula is

$$\frac{\partial M^{(k)}_\alpha}{\partial V(\lambda)} = (\ldots)\,\frac{\partial\mu_\alpha}{\partial V(\lambda)} + \sum_{\beta\neq\alpha}(\ldots)\,\frac{\partial\mu_\beta}{\partial V(\lambda)} - \frac{\partial}{\partial\lambda}\left(\frac{1}{(\lambda-\mu_\alpha)^k\tilde{y}(\lambda)}\right), \tag{148}$$

In order to calculate the derivative of the branching point μ_α w.r.t. the potential, one can note that $W_0(\lambda,\mu) = \frac{1}{2}\frac{\partial(V'(\mu)-y(\mu))}{\partial V(\lambda)}$ and bring the variable μ in this expression to μ_α. Then, from (97) and (101), one obtains that

$$M^{(1)}_\alpha\frac{\partial\mu_\alpha}{\partial V(\lambda)} = \frac{1}{(\lambda-\mu_\alpha)\tilde{y}(\lambda)} - \sum_{i=1}^{n-1}\frac{H_i(\lambda)}{\tilde{y}(\lambda)}\oint_{A_i}\frac{d\xi}{(\xi-\mu_\alpha)\tilde{y}(\xi)}. \tag{149}$$

It immediately follows from these formulas and (100), (101) that integrals over A-cycles of both $\frac{\partial\mu_\alpha}{\partial V(\lambda)}$ and $\frac{\partial M^{(k)}_\alpha}{\partial V(\lambda)}$ vanish and one, therefore, arrives at (137).

Using the above conditions and formula (137), we can now invert the operator $\widehat{K}-2W_0(\lambda)$ when acting on basis monomials $(\lambda-\mu_\alpha)^{-k}$. That is, we are going to check that the basis $\chi^{(k)}_\alpha(\lambda)$ vectors (145) are generated from these basis monomials by the operator $\widehat{d\mathcal{G}}$

$$\chi^{(k)}_\alpha(\lambda) = \oint_{\mathcal{C}_{\mu_\alpha}}\frac{d\mu}{2\pi i}\frac{1}{y(\mu)}\frac{d\mathcal{G}(\lambda,\mu)}{d\lambda}\cdot\frac{1}{(\mu-\mu_\alpha)^k} \equiv \widehat{d\mathcal{G}}\left((\lambda-\mu_\alpha)^{-k}\right), \tag{150}$$

First few basis functions are easy to obtain from (149) and (139):

$$\begin{aligned}
\chi^{(1)}_\alpha(\lambda) &= \frac{\partial\mu_\alpha}{\partial V(\lambda)}, \quad \alpha=1,\ldots,2n,\\
\chi^{(2)}_\alpha(\lambda) &= -\frac{2}{3}\frac{\partial}{\partial V(\lambda)}\log|M^{(1)}_\alpha| -\\
&\qquad -\frac{1}{3}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{2n}\frac{\partial}{\partial V(\lambda)}\log|\mu_\alpha-\mu_\beta|.
\end{aligned}\tag{151}$$

[21]This can be also stated from the analysis of the loop equations as above.

Proof of (138). First, let us demonstrate that the action of the operator $\widehat{d\mathcal{G}}$ defined in (150) on any basis function $(\lambda-\mu_\alpha)^{-k}$ inverts the action of $\widehat{K}-2W_0(\lambda)$ up to the zero mode content. For this, let us consider the expression

$$
\begin{aligned}
&(\widehat{K}-2W_0(\lambda))\oint_{\mathcal{C}_\mathcal{D}}\frac{d\mu}{2\pi i}\frac{d\mathcal{G}(\lambda,\mu)}{y(\mu)(\mu-\mu_\alpha)^k}\\
&= \oint_{\mathcal{C}_{\mathcal{D}_w}}\frac{dw}{2\pi i}\frac{V'(w)}{\lambda-w}\oint_{\mathcal{C}_{\mathcal{D}_\mu}}\frac{d\mu}{2\pi i}\frac{d\mathcal{G}(w,\mu)}{dw}\frac{1}{y(\mu)(\mu-\mu_\alpha)^k}-\\
&\qquad -2W_0(\lambda)\oint_{\mathcal{C}_{\mathcal{D}_\mu}}\frac{d\mu}{2\pi i}\frac{d\mathcal{G}(\lambda,\mu)}{d\lambda}\frac{1}{y(\mu)(\mu-\mu_\alpha)^k}.
\end{aligned}
$$

Taking into account that the contour ordering is such that $\mathcal{C}_{\mathcal{D}_w}>\mathcal{C}_{\mathcal{D}_\mu}$ in the sense that one lies inside the other and evaluating the integral over w by taking residues at the points $w=\lambda$ and $w=\infty$, we find that the result of the residue at $w=\lambda$ combines with the second term to produce $V'(\lambda)-W_0(\lambda)=y(\lambda)$ while we can replace $V'(w)$ by $y(w)$ when evaluating the residue at infinity due to the asymptotic conditions. That is, we obtain

$$
\begin{aligned}
&y(\lambda)\oint_{\mathcal{C}_{\mathcal{D}_\mu}}\frac{d\mu}{2\pi i}\frac{d\mathcal{G}(\lambda,\mu)}{d\lambda}\frac{1}{y(\mu)(\mu-\mu_\alpha)^k}+\oint_{\mathcal{C}_\infty}\frac{dw}{2\pi i}\frac{y(w)}{\lambda-w}\oint_{\mathcal{C}_{\mathcal{D}_\mu}}\frac{d\mu}{2\pi i}\frac{d\mathcal{G}(w,\mu)}{dw}\frac{1}{y(\mu)(\mu-\mu_\alpha)^k}\\
&= \oint_{\mathcal{C}_{\mathcal{D}_w}}\frac{dw}{2\pi i}\frac{y(w)}{\lambda-w}\oint_{\mathcal{C}_{\mathcal{D}_\mu}}\frac{d\mu}{2\pi i}\frac{d\mathcal{G}(w,\mu)}{dw}\frac{1}{y(\mu)(\mu-\mu_\alpha)^k},
\end{aligned}
$$

where the contour ordering is such that $\lambda>\mathcal{C}_{\mathcal{D}_w}>\mathcal{C}_{\mathcal{D}_\mu}$. We now want to push the integration contour for μ through the integration contour for w. After it, the obtained integral over μ vanishes as the integrand is then analytic everywhere outside $\mathcal{C}_{\mathcal{D}_\mu}$. Thus, contributions come only from the pole at $w=\mu$ of $d\mathcal{G}(w,\mu)$, which contributes when pushing the contour $\mathcal{C}_{\mathcal{D}_\mu}$ through $\mathcal{C}_{\mathcal{D}_w}$, and from the multiple-valuedness of $d\mathcal{G}(w,\mu)$ w.r.t. the variable μ. Note, however, that these latter contributions are always proportional to $d\omega_i(w)=\frac{H_i(w)d\lambda}{\tilde{y}(w)}$. That is, we have

$$
\begin{aligned}
&\oint_{\mathcal{C}_{\mathcal{D}_w}}\frac{dw}{2\pi i}\frac{y(w)}{\lambda-w}\frac{1}{y(w)(w-\mu_\alpha)^k}\\
&\quad+\mathrm{const}\cdot\oint\oint_{\lambda>\mathcal{C}_{\mathcal{D}_\mu}>\mathcal{C}_{\mathcal{D}_w}}\frac{dw}{2\pi i}\frac{d\mu}{2\pi i}\frac{y(w)}{\lambda-w}\frac{H_i(w)}{\tilde{y}(w)}\frac{1}{y(\mu)(\mu-\mu_\alpha)^k},
\end{aligned}
$$

where the point λ lies outside the integration contour in the first term and the integral over w in the last term vanishes because the integrand

$$
\frac{y(w)H_i(w)}{(\lambda-w)\tilde{y}(w)}=\frac{M(w)H_i(w)}{\lambda-w}
$$

is obviously regular everywhere inside the contour $\mathcal{C}_{\mathcal{D}_w}$ (recall that both λ and μ are now outside this contour). Upon integration, the first term obviously produces $(\lambda-\mu_\alpha)^{-k}$, which completes the first part of the proof.

Next, note that conditions (137) hold automatically for any function $f(\lambda)$, having singularities only at μ_α, transformed by the operator $\widehat{d\mathcal{G}}$, (140) due to the normalization properties of the kernel $d\mathcal{G}(\lambda,\mu)$ (139). Therefore, the result of the action of this operator can be always presented as the linear combination of the functions $\frac{\partial\mu_\alpha}{\partial V(\lambda)}$ and $\frac{\partial M_\alpha^{(k)}}{\partial V(\lambda)}$, which completes the proof of formula (150).

4.2 Calculations in genus one

Now we invert the loop equations for genus $h=1$ and integrate them to obtain the genus one free energy. We need, in this case, only $\chi_\alpha^{(1)}(\lambda)$ and $\chi_\alpha^{(2)}(\lambda)$ (see (146)) which we already have, (151), and

eq.(11) reads

$$(\widehat{K} - 2W_0(\lambda))W_1(\lambda) = \frac{\partial}{\partial V(\lambda)} W_0(\lambda). \tag{152}$$

Given $W_0(\lambda)$ (26), the r.h.s. becomes

$$\begin{aligned}
\frac{\partial}{\partial V(\lambda)} W_0(\lambda) &= -\frac{3}{16}\sum_{\alpha=1}^{2n}\frac{1}{(\lambda-\mu_\alpha)^2} - \frac{1}{8}\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{2n}\frac{1}{(\lambda-\mu_\alpha)(\lambda-\mu_\beta)} \\
&\quad + \frac{1}{4}\tilde{y}(\lambda)\sum_{\alpha=1}^{2n}\frac{1}{\lambda-\mu_\alpha}M_\alpha^{(1)}\frac{\partial \mu_\alpha}{\partial V(\lambda)} \\
&= \frac{1}{16}\sum_{\alpha=1}^{2n}\frac{1}{(\lambda-\mu_\alpha)^2} - \frac{1}{8}\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{2n}\frac{1}{\mu_\alpha-\mu_\beta}\left(\frac{1}{\lambda-\mu_\alpha} - \frac{1}{\lambda-\mu_\beta}\right) \\
&\quad + \frac{1}{4}\sum_{\alpha=1}^{2n}\frac{\mathcal{L}_\alpha(\mu_\alpha)}{\lambda-\mu_\alpha}.
\end{aligned} \tag{153}$$

Here we took into account that regular parts coming from $\frac{\lambda^l}{\lambda-\mu_\alpha}$, $l = 1, \ldots, n-2$, vanish for $W_0(\lambda,\lambda) = \frac{\partial}{\partial V(\lambda)}W_0(\lambda)$ to satisfy the correct asymptotic behavior, and we can just replace λ^l by μ_α^l in numerators of such expressions. The result for the one-point resolvent of genus one with n cuts can now be easily obtained using Eq. (151),

$$\begin{aligned}
W_1(\lambda) &= \frac{1}{16}\sum_{\alpha=1}^{2n}\chi_\alpha^{(2)}(\lambda) - \frac{1}{8}\sum_{1\le\alpha<\beta\le 2n}\frac{1}{\mu_\alpha-\mu_\beta}\left(\chi_\alpha^{(1)}(\lambda) - \chi_\beta^{(1)}(\lambda)\right) \\
&\quad + \frac{1}{4}\sum_{\alpha=1}^{2n}\mathcal{L}_\alpha(\mu_\alpha)\chi_\alpha^{(1)}(\lambda) \\
&= \frac{1}{16}\sum_{\alpha=1}^{2n}\left(-\frac{2}{3}\frac{\partial}{\partial V(\lambda)}\log|M_\alpha^{(1)}| - \frac{1}{3}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{2n}\frac{\partial}{\partial V(\lambda)}\log|\mu_\alpha-\mu_\beta|\right) \\
&\quad - \frac{1}{8}\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{2n}\frac{1}{\mu_\alpha-\mu_\beta}\left(\frac{\partial\mu_\alpha}{\partial V(\lambda)} - \frac{\partial\mu_\beta}{\partial V(\lambda)}\right) \\
&\quad + \frac{1}{4}\sum_{\alpha=1}^{2n}\mathcal{L}_\alpha(\mu_\alpha)\frac{\partial\mu_\alpha}{\partial V(\lambda)}.
\end{aligned} \tag{154}$$

Now one should integrate (154) in order to obtain $\mathcal{F}_1$. While integrating the first two terms in the r.h.s. is straightforward, the term with the zero modes requires some more work. Using formulas (103) and (105), we see that the last term in (154) is just

$$-\frac{1}{2}\sum_{\alpha=1}^{2n}\frac{\partial}{\partial\mu_\alpha}\left(\log\det_{i,j=1,\ldots,n-1}\sigma_{j,i}\right)\frac{\partial\mu_\alpha}{\partial V(\lambda)},$$

i.e.,

$$\mathcal{F}_1 = -\frac{1}{24}\log\left(\prod_{\alpha=1}^{2n}M(\mu_\alpha)\cdot\Delta^4\cdot\left(\det_{i,j=1,\ldots,n-1}\sigma_{j,i}\right)^{12}\right), \tag{155}$$

where $\Delta = \prod_{1\le\alpha<\beta\le 2n}(\mu_\alpha-\mu_\beta)$ is the Vandermonde determinant. This is our final answer for the genus-one partition function[22].

[22] One has to compare this answer with the formula proposed in [56, 24, 57] for the one-loop (toric) corrections in topological theories,

$$\mathcal{F}_1(t_I) = \frac{1}{24}\log\det\left[\frac{\partial^3\mathcal{F}_0}{\partial t_\alpha\partial t_\beta\partial t_\gamma}\partial_X t_\gamma\right] + G(\{t_\alpha\}).$$

4.3 Genus one free energy and determinant representation

Let us now discuss the expression for the genus one free energy. First of all, notice that (155) reproduces the calculation of [34] for the two-cut solution, up to a modular transformation permuting $A-$ with $B-$ cycles. This should be a surprise, since we put throughout our calculation in sect. 4.2 the constraint that A-periods (30) of the generating differential (67) are constant under the action of the operator $\frac{\partial}{\partial V(p)}$, see (72). On the contrary, in [34] Akemann imposed the condition of vanishing B-periods of dS, corresponding to equal "levels" Π_i in different wells of the potential [58] or additional minimization of the free energy (39) at the saddle point w.r.t. the occupation numbers (41). In fact, since neither the answer (155) nor intermediate calculations contain any manifest dependence on particular values of the periods, one can equally put all B-periods (during the calculation of [34]) fixed to be arbitrary non-vanishing constants and, therefore, come to a modular transformed counterpart of our choice of the normalizing cycles.

Under condition of constant B-periods (44)-(47), as it was stressed in [23], the matrix $\sigma_{i,j}$ of the A-periods of $\frac{x^i dx}{\tilde{y}(x)}$ is replaced by the matrix of the corresponding B-periods. This is the only difference with the result of [34]; certainly formula (155) reproduces the answer of [51, 52][23] for generic multi-cut solution.

The fact, that the only result of interchanging A- and B-cycles is the interchanging of the corresponding periods in $\det\sigma$ implies that $e^{\mathcal{F}_1}$ is a *density* and not a scalar function on moduli space of the curves. Indeed, when exchanging A- and B-cycles, $\det\sigma_{i,j}$ is multiplied by $\det\tau_{ij}$ – the determinant of the period matrix of the curve. In order to compensate this factor, $e^{\mathcal{F}_1}$ must be transformed under such transformation with the additional factor $(\det\tau_{ij})^{1/2}$, as follows from (155). Then one immediately comes to the above observation: exchange of A- and B-cycles results only in replacing the corresponding matrix $\sigma_{i,j}$ in (155). Note that such behavior of $e^{\mathcal{F}_1}$ indicates that it is a section of determinant bundle DET$\bar{\partial}$ over the moduli space, where the $\bar{\partial}$-operator acts on the sections of a non-trivial bundle on a complex curve of matrix model. One can find that determinant $\det'\bar{\partial}_j$ of the $\bar{\partial}$-operator (with some fixed basis of the zero modes), acting on j-differentials, is proportional to $\det\sigma$ for $j=0,1$ but for other values of j it typically does not contain the factor $\det\sigma$, still transforming non-trivially under exchange of A- and B-cycles. It was proposed in [52] that, in order to match the proper behavior under modular transformations, the operator $\bar{\partial}_j$ should act on twisted bosons on hyperelliptic curves, then $e^{\mathcal{F}_1}$ actually equals to its determinant. Besides, one also needs to add some corrections from the star operators [52, 54] that do not contain $\det\sigma$ factors and cannot be restored by modular covariance of the answer; these are necessary to obtain the correct result (155).

There is another important point that differs between our formula and the result of [34]. Namely, while in [34] it was possible to add any constant to the final result, not spoiling the solution to the loop equation, we can add to (155) an arbitrary *function* of occupation numbers S_i, since in contrast to [34] with no free parameters, we keep S_i's arbitrary.

One can partially fix this arbitrary function in the free energy by imposing requirement of smooth behavior of $\mathcal{F}_1$ under degenerations of the surface. To this end, let us shrink one of the cuts, e.g. bring μ_2 to μ_1. Setting $\mu_2-\mu_1=\epsilon\to 0$, we can easily check that

$$\mathcal{F}_1^{(n)} \sim -\frac{1}{24}\log\left[\epsilon^4\prod_{\alpha=3}^{2n}(\mu_1-\mu_\alpha)\prod_{i=1}^{m-n}(\mu_1-\lambda_i)^2\right]+\mathcal{F}_1^{(n-1)}+O(\epsilon) \tag{156}$$

In order to compensate the first term that spoils the smooth degeneration, one suffices to add

$$+\frac{1}{12}\prod_{i=1}^{n}\log S_i, \qquad S_n\equiv t_0-\sum_{i=1}^{n-1}S_i \tag{157}$$

Note that this term is out of control in the conformal field theory approach of [51, 52]. On the other hand, it could be also compared with the matrix model calculations of [6, 23]. An arbitrary function

[23] If restoring in [51] the determinant term $\det\sigma$, omitted from the answer.

of S_i comes there from different normalizations of the matrix integral. In particular, the normalization in [6, 23] corresponds just to (155) without adding (157).

This is, in fact, a general phenomenon in the matrix model calculations: for any genus the only source for the singular contribution comes from degenerate geometry of curves and is related with normalization factor in the matrix integral, that is, the volume of (the orbit of) the unitary group. Indeed, the integral itself is a Taylor series in S_i's (see formula (4.8) in [23]), while the unitary group volume [59] contributes with the factor

$$\prod_i \left(\prod_{l=1}^{S_i/\hbar} \Gamma(l) \right) \equiv \prod_i G_2(S_i/\hbar) \tag{158}$$

where $G_2(x)$ is the Barns function, [60]. Now using the asymptotic expansion for the Γ-function at large values of argument and formulas relating the Γ-function and the Barns function (see, e.g., [61]), one finds the asymptotic expansion of the Barns function [60, 62]

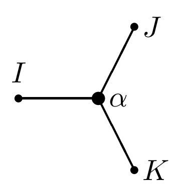
$$G_2(N) = \log \left(\prod_{l=1}^{N} \Gamma(l) \right) = \frac{S^2}{2} \log N - \frac{1}{12} \log N - \frac{3}{4} N^2 + \frac{1}{2} N \log 2\pi + \zeta'(-1) + \sum_{h=2} \frac{B_{2h}}{4h(h-1)} \frac{1}{N^{2h-2}} \tag{159}$$

where B_{2h} are Bernoulli coefficients and $\zeta(s)$ is the Riemann ζ-function. Thus, one obtains that the singular contribution in genus h is [23]

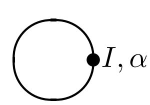
$$\mathcal{F}_h = \sum_i^n \frac{B_{2h}}{4h(h-1)} \frac{1}{S_i^{2h-2}} \qquad h \geq 2 \tag{160}$$

This is the simplest way to pick up the singular contribution, although it can be also done, genus by genus, by direct solving the loop equations, like it was demonstrated above for the genus one case.

Now let us turn to another important issue. In sect. 3, we obtained formula (129), expressing the third derivatives of $\mathcal{F}_0$ through the quantities $\phi_{I\alpha}$ determined in (128). Since one often interprets $\mathcal{F}$ as the free energy of a topological string theory, one could naturally associate the third derivative of $\mathcal{F}_0$ with the tree three-point function in this theory, i.e. represent (129) as three "propagators" $\phi_{I\alpha}$ ending at the same "3-vertex":



In such case, one has to associate $\mathcal{F}_1$ with the one-loop diagram in this topological theory, i.e. with the propagator determinant $\det_{I,\alpha} \phi_{I\alpha}$:



Calculating this determinant stems actually to calculating the polynomial determinant $\det_{I,\alpha} H_I(\mu_\alpha)$. We already saw that, due to normalization conditions (57) and (74), the polynomials $H_K(\lambda)$ corresponding to the variables t_k with $k > 0$ always have the coefficient k at the highest term λ^{n-1+k} while the polynomial $H_0(\lambda)$ starts with unit coefficient at λ^{n-1}. Passing from $H_i(\lambda)$, corresponding to the variables S_i, $i = 1, \dots, n-1$, to the basis of monomials λ^{i-1}, one obtains that the total determinant

is then (up to a trivial factor $n!$) just the total Vandermonde determinant divided by the determinant of the transition matrix σ (104). The complete answer for the determinant of $\phi_{I\alpha}$ is then

$$\det_{I,\alpha} \phi_{I\alpha} = \left(\prod_{\alpha=1}^{2n} M_{m-n}(\mu_\alpha) \right)^{-1/3} \Delta(\mu)^{-1/3} (\det \sigma)^{-1} \tag{161}$$

and the second of matching conditions (134) stems now to the condition of nontriviality of $\det \sigma$.[24]

Comparing (161) and (155), one finds that powers of $\det \sigma$ and of the Vandermonde determinant $\Delta(\mu)$ in these expressions perfectly match, i.e. $\mathcal{F}_1$ is indeed proportional to the determinant, up to non-universal pieces containing M and (arbitrary function of) S_i. These pieces remain due to the freedom in defining the measure in the path integral.

Therefore, we conjecture an existence of a diagram technique for calculating the higher genera free energy and/or generating function for the correlators[25]. Would such a diagram technique be constructed in full, it opens a possibility of calculating the higher genera/multi-point contributions in a rather effective way. Therefore, it would be of great practical use to make further checks of the conjecture.

4.4 Relation to topological B-model

The authors of [23] proposed an anzatz for $\mathcal{F}_1$ in the two-cut case (with absent double points). Their formula in fact comes from the correspondence between the so called topological B-model on the local Calabi-Yau geometry $\widehat{II}$ and the cubic matrix model conjectured in [3]. However, this does not completely fix the formula for $\mathcal{F}_1$, leaving room for a certain holomorphic ambiguity, which was fixed in [23] basically by some simplicity arguments.

First of all, introduce the quantities $\mu^-_{1,2} \equiv \{\mu_2 - \mu_1, \mu_4 - \mu_3\}$, i.e. complexified lengths of the two cuts on hyperelliptic plane, and $\{S_1, S_2 \equiv S_n = t_0 - S_1\}$. Then, one expects

$$\mathcal{F}_1 = \frac{1}{2} \log \left(\det \left\| \frac{\partial \mu^-_{1,2}}{\partial S_{1,2}} \right\| \Delta(\mu)^{2/3} (\mu_1 + \mu_2 - \mu_3 - \mu_4)^{-1} \right). \tag{162}$$

This formula was, indeed, checked for a few first terms of expansion in S_i's [23] and it is proven by the direct calculation in [64].

Below we propose a similar formula for the case of any number of cuts (see also the details in [64]). Let us divide all the branching points into two ordered sets $\{\mu^{(1)}_j\}^n_{j=1}$ and $\{\mu^{(2)}_j\}^n_{j=1}$ and perform then a linear orthogonal transformation of $\mu^{(1,2)}_j$ to the quantities $\{\mu^+_j\}^n_{j=1}$ and $\{\mu^-_j\}^n_{j=1}$ by

$$\mu^\pm_j = \mu^{(1)}_j \pm \mu^{(2)}_j. \tag{163}$$

Taking now $n-1$ canonical variables S_i, the variable $S_n = t_0 - \sum_{i=1}^{n-1} S_i$, p *lower* times t_k, $k = 1, \dots, p$ $(0 \le p \le n)$, and choosing an *arbitrary* set of $n+p$ branching points μ_{α_j}, $j = 1, \dots, n+p$, following

[24]This is easy to prove. Would be $\det \sigma = 0$, one obtains that there must exist a polynomial $P(\lambda)$ of degree less or equal $n-2$ such that

$$\int_{\lambda_{2i-1}}^{\lambda_{2i}} \frac{P(\lambda) d\lambda}{\tilde{y}} = 0 \quad \text{for} \quad i = 1, \dots, n-1.$$

This necessarily implies that $P(\lambda)$ has at least one zero at each of the intervals $(\lambda_{2i-1}, \lambda_{2i})$; otherwise the combination under the integral sign is sign definite and the integral cannot vanish. The polynomial $P(\lambda)$ must then have at least $n-1$ zero and, having the degree not exceeding $n-2$, must therefore vanish.

[25]Note that this conjectured diagram technique is different from that of [32]. Note also that recent paper [63], where the diagram technique of [32] was extended from calculating resolvents to the free energy calculations, possesses a clear disadvantage: intermediate patterns appeared are manifestly non-symmetric w.r.t. field propagators.

the same logic as for (161) (see also (126)), we obtain

$$\det\left\|\frac{\partial\{\mu_{\alpha_j}\}}{\partial\{S_i,S_n,t_k\}}\right\| = \frac{\Delta(\mu_{\alpha_j})\cdot(\det\sigma)^{-1}}{\prod_{j=1}^{n+p} M^{(1)}_{\alpha_j}\prod_{j=1}^{n+p}\left(\prod_{\beta\neq\alpha_j}^{2n}(\mu_{\alpha_j}-\mu_\beta)\right)} \tag{164}$$

with the *same* matrix σ for any choice of the set of indices $\{\alpha_j\}_{j=1}^{n+p}$ and any number p of canonical times t_k (but only for $0\le p\le n$). Set all $M^{(1)}_\alpha\equiv 1$; the Vandermonde determinant $\Delta(\mu_{\alpha_j})$ then combines with the rational factors in the denominator to produce $(-1)^{\sum_{j=1}^n \alpha_j}\Delta(\overline{\mu_{\alpha_j}})/\Delta(\mu)$, where $\Delta(\overline{\mu_{\alpha_j}})$ is the Vandermonde determinant for the supplementary set of $n-p$ branching points not entering the set $\{\mu_{\alpha_j}\}_{j=1}^{n+p}$ whereas $\Delta(\mu)$ is the total Vandermonde determinant. In particular, when $p=0$, splitting μ_α as in (163) and using formulas (155) and (164), we have

$$\mathcal{F}_1\Bigg|_{M^{(1)}_\alpha\equiv 1} = \frac{1}{2}\log\left(\det\left\|\frac{\partial\{\mu_j^-\}}{\partial\{S_i,S_n\}}\right\|\Delta(\mu)^{2/3}\Delta^{-1}(\mu_j^+)\right), \tag{165}$$

where the additional Vandermonde determinant is taken w.r.t. the supplementary variables μ_j^+. In the two-cut case it reproduces (162).

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